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Some functional equations which generate both crinkly broken lines and curves

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§0. Preliminary.

Since the discovery of Peano's plane filling curve, we have known many irregular curves as von Koch curve [5], Pólya curve [11] and Lévy curve [6]. In recent years, these curves have found to be related to Mandelbrot's Fractal theory and studied by many authors. They have shown that these curves can be defined as limits of sequences of broken lines generated by a kind of transformation. In fact, such an idea has already given in [9] by E. H. Moore.

A sequence of complex numbers $a=\{a_n\}_{n=0}^{+\infty}\in \boldsymbol{C}^{\boldsymbol{N}}$ generates a broken line $L(a)$ in the comprex plane whose turning points $\{z_n\}_{n=0}^{+\infty}$ are given by

$$z_0=0, \quad z_n=\sum_{k=0}^{n-1} a_k \quad n=1, 2,\dots.$$

Definition. For $(\gamma_0, \gamma_1,\dots, \gamma_{n-1})\in \boldsymbol{C}^n-\{0\}$, we define T_γ: $\boldsymbol{C}^{\boldsymbol{N}}\rightarrow \boldsymbol{C}^{\boldsymbol{N}}$ such that

$$T_\gamma(a_0, a_1,\dots)=(\gamma_0 a_0, \gamma_1 a_0,\dots, \gamma_{n-1}a_0, \gamma_0 a_1, \gamma_1 a_1,\dots).$$

This transformation T_γ replaces a segment a_j by segments $\gamma_0 a_j$, $\gamma_1 a_j$, $\gamma_2 a_j,\dots,$ $\gamma_{n-1}a_j$. We will treat such a type of transformation in a new view point, which is different from previous studies, in the following sections. In §1, we will direct our attention to the broken lines which are invariant under T_γ. Roughly speaking, the invariant broken line is the eigen vector of the linear map T_γ and in the course of the discussion, the eigen vector turns out to be the solution of a functional equation on the formal power series $\boldsymbol{C}[[z]]$. In §2, it will be also shown that the solution of the functional equation has the natural boundary at the unit circle. In the rest of this section, we will review the previous results in our formulation.

The following theorem gives the process of generation of the irregular curves by T.

Theorem 1. *Let $a=\{a_j\}\in \boldsymbol{C}^{\boldsymbol{N}}$ and $\gamma=(\gamma_0,\dots, \gamma_{n-1})\in \boldsymbol{C}^n$. If $\sum_{j=0}^{n-1}\gamma_j=1$, $|\gamma_j|<1$ for all $j=0, 1,\dots, n-1$ and $L(a)$ is a compact subset of the plane, then $L(T_\gamma^n(a))$ is compact and converges as $n\rightarrow+\infty$ in the sense of Hausdorff metric.*

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