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On boundary behaviours of holomorphic functions

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1. Introduction.

Let $D=\{z: |z|<1\}$ and let $w=S(z)$ denote an arbitrary one-to-one conformal mapping from D onto itself. A function $f(z)$, meromorphic in D, is said to be normal in D [12, p. 53], if the family of functions $\{f(S(z))\}$ is normal in D in the sense of Montel, where convergence is defined in terms of the spherical metric. Following Bagemihl and Seidel [3, p. 10], we call $e^{i\theta}$ a Fatou point of f if f has an angular limit v (possibly ∞) at $e^{i\theta}$. This limit v will be called a Fatou value of f.

There is a normal meromorphic function in D which possesses no Fatou points [12, p. 58]. On the other hand, Bagemihl and Seidel proved that any normal holomorphic function f in D must have Fatou points everywhere dense on the circle $C=\{z: |z|=1\}$ [3, Corollary 1]. In fact, in [3, Theorem 3], they proved that if the set of Fatou points of f on an arc Γ of C is of measure zero, then the arc Γ contains a Fatou point at which the corresponding Fatou value is ∞. Moreover, they constructed a normal holomorphic function f for which the measure of the set of all Fatou points of f is less than any prescribed small number and the function f has no infinite Fatou values [3, Theorem 4]. They then asked that if f is normal holomorphic in D and if there is an arc Γ on C such that the measure of the set of Fatou points of f on every subarc of Γ is less than the length of that subarc, does the arc Γ contain a Fatou point of f whose Fatou value is ∞? The answer turns out to be negative due to S. Dragosh [6].

A problem related to the above one was asked by MacLane [13]. Following MacLane [13, p. 8], we denote by $\mathscr{A}$ the class of all non-constant holomorphic functions f in D such that f has asymptotic values on a set S which is dense in C, namely, for each $p\in S$, there is a Jordan arc J lying in D and tending to p such that $f(z)$ tends to a value along J. In contrast to the notion of Fatou points, we shall call a point $p\in C$ an asymptotic point of f if f has an asymptotic value at p. Let A be the set of all asymptotic points of f on C and let A_∞ be the subset of A containing all points of A with the asymptotic value ∞. In [13, p. 77], MacLane asked that if $f\in\mathscr{A}$ and if Γ is an arc on C with $\Gamma\cap A_\infty=\emptyset$, is it true that the arc Γ contains a subarc γ such that almost every point of γ is an asymptotic point of f. Again, the answer is negative due to Dragosh (He has not mentioned this assertion in [6]).

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