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# A note on the function $\sum_{n=1}^{\infty} [nx+y]/n!$

By

Iekata SHIOKAWA and Jun-ichi TAMURA

Let $f(x, y)$ be the function of real variables $x$ and $y$ defined by

$$f(x, y)=\sum_{n=1}^{\infty}\frac{[nx+y]}{n!},$$

where $[t]$ denotes the greatest integer not exceeding the real number $t$. In this paper we prove in §3 the linear independency over the filed $Q$ of all rationals of the values of $f(x, y)$ for different irrationals $x$ and in §2 their transcendency for rationals $x$. Also some properties of the function $f(x, y)$ are studied in §1.

## 1. Some properties of the function $f(x, y)$.

From the definition it follows that

$$f(x, y)=e[x]+(e-1)[y]+f(\{x\}, \{y\}), \tag{1}$$

where $\{t\}=t-[t]$. It is easily seen that

$$f(x, y)\neq f(x', y') \quad \text{if} \quad (x, y)\neq(x', y'),$$

except when $x=x'$ is a rational number, say $x=p/q$ with coprime integers $q>0$ and $p$, and $r/q\leqq y, y'<(r+1)/q$ for some integer $r$; in this special case we have

$$f(p/q, y)=f(p/q, r/q) \quad \text{if} \quad r/q\leqq y<(r+1)/q. \tag{2}$$

The quantity in the right-hand side of (2) will be expressed in Theorem 1 as a linear form of the values of the exponential function. If $x$ is an irrational number, $f(x, y)$ is strictly increasing as a function of $y$. $f(x, y)$ is also strictly increasing as a function of $x$ for any fixed $y$, not necessarily irrational.

The function $[x]$ satisfies the equality

$$[nx]=\sum_{r=0}^{q-1}\left[\frac{nx}{q}+\frac{r}{q}\right]$$

for any positive integer $q$, so that we find

$$f(x, 0)=\sum_{r=0}^{q-1} f\left(\frac{x}{q}, \frac{r}{q}\right),$$

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