J. Math. Kyoto Univ. (JMKYAZ)
27-1 (1987) 47-59

Remarks on null solutions of linear partial differential equations

By

Katsuju IGARI

Introduction.

Null solution. Let $P(x; \partial_x)$ be a linear partial differential operator of order m defined in a neighborhood of the origin in $\boldsymbol{R}^d$. Let $\varphi(x)$ be a real-valued function such that $\varphi(0)=0$ and $\varphi_x(0)\neq 0$. Let S stand for the hypersurface defined by $\varphi(x)=0$. We assume S is characteristic to $P(x; \partial_x)$, i.e., $P_m(x; \varphi_x(x))=0$ on S. Here P_m denotes the principal part of P.

We call a solution u of $Pu=0$ a null solution if $\{0\}\in \mathrm{supp}\,[u]\subset\{x;\ \varphi(x)\geqq 0\}$. We are concerned, in the present paper, with the question firstly raised by Petrowski whether there exists a null solution of $Pu=0$. When all the coefficients of P and $\varphi(x)$ are analytic, the question is related to the inverse of Holmgren's uniqueness theorem. Since null solution is non-analytic at S, the existence of null solution implies also that the operator is not analytic hypo-elliptic.

Multiplicity. By the way, we defined the multiplicity of characteristic hypersurface, [5]. Let $x\in S$,

$$\Lambda_x=\{(\alpha, \beta);\ P^{(\alpha)}_{m(\beta)}(x, \varphi_x(x))\neq 0\},$$

$$k=\min\{|\alpha|+|\beta|\}, \qquad \text{for} \quad (\alpha, \beta)\in\Lambda_x$$

$$l=\min|\beta|, \qquad \text{for} \quad (\alpha, \beta)\in\Lambda_x\cap\{|\alpha|+|\beta|=k\}.$$

Here $P^{(\alpha)}_{m(\beta)}(x; \xi)=\partial^\beta_x\partial^\alpha_\xi P_m(x; \xi)$, and if Λ_x is empty, we put $k=l=\infty$. We call the pair $(k, l)_x$ the multiplicity of the characteristic hypersurface S at $x\in S$. Evidently, $k\geqq 1$, $0\leqq l\leqq k$, $k-l\leqq m$. This is an invariant notion with respect to the change of variables and also to the choice of $\varphi(x)$, see [5].

Already known facts. Let us assume that all the coefficients of $P(x; \partial_x)$ and $\varphi(x)$ are analytic and the multiplicity (k, l) of characteristic hypersurface S is constant on S itself. If $l<k$, whatever the lower order terms are, there exists a C^∞ null solution which is analytic for $x\notin S$. This fundamental theorem was proved by S. Ouchi [10] preceded by the works of S. Mizohata [8], L. Hörmander [2], J. Persson [12], H. Komatsu [7], and so on. However, when $l=k$, the question seems to take a different aspect. Among such operators there are Fuchs type ones

Communicated by Prof. S. Mizohata June 8, 1985, Revised October 5, 1985