J. Math. Kyoto Univ. (JMKYAZ)
28-1 (1988) 133-139

# On the $L^2$-boundedness of pseudo-differential operators

By

Yoshiki HIGUCHI and Michihiro NAGASE

## §1. Introduction.

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. Let $m$, $\rho$ and $\delta$ be real numbers with $0\leq\rho$, $\delta\leq 1$. If a smooth function $p(x, \xi)$ on $\mathbf{R}^n_x\times\mathbf{R}^n_\xi$ satisfies

$$(1.1)\qquad |\partial_\xi^\alpha\partial_x^\beta p(x, \xi)|\leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for any multi-indices $\alpha$ and $\beta$, then we say that $p(x, \xi)$ belongs to Hörmander's class $S^m_{\rho,\delta}$ (see, for example, [5]). For $p(x, \xi)$ in $S^m_{\rho,\delta}$ we define the pseudo-differential operator $p(X, D_x)$ by

$$(1.2)\qquad p(X, D_x)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is, $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$ and we denote $p(X, D_x)\in L^m_{\rho,\delta}$. The function $p(x, \xi)$ is called the symbol of the operator $p(X, D_x)$. In [4], Hörmander proved that if all of the operators in $L^m_{\rho,\delta}$ are $L^2(\mathbf{R}^n)$-bounded then $m\leq\min\left\{0, \frac{n}{2}(\rho-\delta)\right\}$, by giving counter examples. When $\delta<1$, Calderón and Vaillancourt in [1] showed that $m\leq\min\left\{0, \frac{n}{2}(\rho-\delta)\right\}$ implies the $L^2(\mathbf{R}^n)$-boundedness of the operators in $L^m_{\rho,\delta}$. Moreover, when $\rho=\delta<1$, there are many generalized theorems to the case of non-regular symbols (see, for example, [3], [7] and [12]). On the other hand, when $\delta=1$, Chin-Hung-Ching in [2] proved that $S^0_{1,1}$ does not always define $L^2(\mathbf{R}^n)$-bounded operators, and Rodino in [11] proved that the operator in $L^{-n(1-\rho)/2}_{\rho,1}$ is not always $L^2(\mathbf{R}^n)$-bounded, by constructing the counter examples.

In the present paper we give also an example of symbols which is in $S^{-n(1-\rho)/2}_{\rho,1}$ but define operators unbounded in $L^2(\mathbf{R}^n)$, and we show that the decreasing order $n(1-\rho)/2$ of symbols is critical in a sense (see [6]). Our example is similar to the example constructed by Chin-Hung-Ching in the case $\rho=1$, and therefore a little different from the one of Rodino in [11].

---

Communicated by Prof. S. Mizohata November 18, 1986