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The imbedding theorems for weighted Sobolev spaces

By

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§ 0. Introduction

Let F be a closed set of $\boldsymbol{R}^n$, and let $\Omega=\boldsymbol{R}^n\backslash F$. The purpose of this paper is to study imbedding theorems for weighted Sobolev spaces of three categories $W_\alpha^{k,p}(\Omega)$, $\mathcal{W}_\alpha^{k,p}(\Omega)$ and $H_\alpha^{k,p}(\boldsymbol{R}^n)$, where the weight functions considered here are powers of $\delta(x)$, where $\delta(x)$ is equivalent to the distance from x to the closed set F, see the § 1 for the precise definitions of those spaces.

In [37–39] S. L. Sobolev introduced a notion of generalized derivative and proved general integral inequalities for differentiable functions of several variables, which are usually lumped together in a single theorem as the so-called Sobolev imbedding theorem. Later the Sobolev theorem was generalized and refined variously (Kondrat'ev, Il'in, Gagliardo, Nirenberg etc.), and such theorems proved to be a usefull tool in functional analysis and in the theory of linear and nonlinear partial differential equations.

Weighted Sobolev spaces also have been studied intensively for more than twenty years, and the main field of application are the degenerated (elliptic) operators. This fact makes clear that a large part of papers concerned with Sobolev spaces with weights where the weight functions are powers of a distance to manifolds (see Grisvard [14], Kufner [20], Lizorkin [23–24], Uspenskii [45] etc.). For the further references, see a survey paper by A. Avantaggiati [8]. The author also studied in [16] the degenerated elliptic operators, and the present paper is strongly motivated by the author's research in this field.

Recently in [30], V. G. Maz'ja has proved a variant of Sobolev imbedding inequalities in the case of a weighted norm in the right-hand side with weights being powers of the distance to a linear subspace of $\boldsymbol{R}^n$, and refined both the Sobolev and Hardy inequalities. Here we present his result as our starting point.

Let us consider functions u of z, where $z=(x, y)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s$ $(1\leq s<n)$. Moreover let μ be a given positive measure on $\boldsymbol{R}^n$ such that the number

$$K=\operatorname*{Sup}_{\rho,z}\,(\rho+|y|)^{-\alpha}\rho^{1-n}[\mu(B_\rho(z))]^{1/q} \tag{0.1}$$

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