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On a Hasse principle for σ-conjugacy

Dedicated to Professor Ichiro Satake on his sixtieth birthday

By

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§ 0. Introduction

Let M/K be a cyclic extension of finite algebraic number fields of degree l, and σ a generator of the Galois group $\mathrm{Gal}(M/K)$, which will be fixed. For an algebraic group G defined over K, we denote by $G(M)$ the set of all points of G with coordinates in M. The action of σ on $G(M)$ can be defined naturally. We denote it by ${}^{\sigma}g$ for $g \in G(M)$. In $G(M)$, we define an equivalence relation $\underset{\sigma}{\sim}$ by $g \underset{\sigma}{\sim} g'$ if and only if $g = h^{-1}g'^{\sigma}h$ for some $h \in G(M)$. This will be called σ-conjugacy. It was introduced for $GL(2)$ in the study of the twisted trace formula ([3], [4]). The purpose of this paper is to determine σ-conjugacy classes for G such that $G(K) = A^{\times}$, where A is a semi-simple algebra over K.

The σ-conjugacy has a close relation with the usual conjugacy, which will be denoted by $\sim$. For $g \in G(M)$, we define the "norm" of $g \in G(M)$ by $Ng = g^{\sigma}g^{\sigma^2}g \cdots {}^{\sigma^{l-1}}g$. Then the conjugacy class of Ng depends only on the σ-conjugacy class of g. We denote by $G(M)/\underset{\sigma}{\sim}$, $G(M)/\sim$ the sets of σ-conjugacy classes, and usual conjugacy classes in $G(M)$ respectively. Then N defines a map of $G(M)/\underset{\sigma}{\sim}$ to $G(M)/\sim$. This map is fundamental in our study of σ-conjugacy. In fact, for $G = A^{\times}$, this map is injective, and to determine $G(M)/\underset{\sigma}{\sim}$, it is sufficient to determine the image of $G(M)/\underset{\sigma}{\sim}$ by N. It is easy to see this image is contained in the set $(G(M)/\sim)^{\sigma}$ consisting of conjugacy classes invariant under σ. To describe the image, we consider the norm at each place of K. For each place v of K, let K_v be the completion of K at v and let $M_v = M \otimes_K K_v$. Then the action of σ can be extended to M_v and $G(M_v)$. We can define in $G(M_v)$ σ-conjugacy and the norm in the same way as above. Our main result asserts that for $G = A^{\times}$ a conjugacy class in $(G(M)/)^{\sigma}$ is contained in the image of $G(M)$ by N if and only if it is contained in image of $G(M_v)$ by N for all v (cf. Th. 2.1).

In § 1, we give preliminary results on σ-conjugacy. In § 2, we state our main result and reduce the proof to the cases of semi-simple and unipotent elements. The proofs of these two cases are given in § 3 and § 4 separately.

§ 1. σ-conjugacy

In this section, we prove some elementary properties of σ-conjugacy. Let K be a field of characteristic 0 and G a linear algebraic group defined over K. We define σ-conjugacy for M more general than that in the Introduction. Let M be a commutative

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