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Simple transcendental extensions of valued fields III: The uniqueness property

By

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Let (K_0, v_0) be a valued field and x be an indeterminate over K_0. For any t in $K_0(x)\setminus K_0$, one can define an extension $v_0{}^t$ of v_0 to a valuation of $K_0(t)$ by:

$$\text{for all } a_0, \cdots, a_n \text{ in } K_0,\ v_0{}^t(a_0+a_1t+\cdots+a_nt^n)=\inf\{v_0(a_i)\,|\,i=0, \cdots, n\}.$$

We are concerned here with the

Uniqueness problem: Given a valuation v of $K_0(x)$ which extends $v_0{}^t$, does there exist a t' in $K_0(x)\setminus K_0$ such that v extends $v_0{}^{t'}$ uniquely?

We proved in [9] that the answer is "yes" if rk v_0 is 1. We shall show here that the answer is also "yes" if v_0 is henselian, and that the answer is "no" in general.

The henselian result, which is proved in section 3, follows from the theorem that, for v_0 henselian, $v_0{}^t$ extends uniquely to a valuation of $K_0(x)$ whenever t has the form $t=f(x)^m/b$, where $f(x)$ is irreducible in $K_0[x]$, b is in K_0, and m is $\geqq 1$.

The negative result is proved in section 2. It follows from the observation that an affirmative answer to the uniqueness problem is equivalent to a fundamental equality, $E=IRD^h$, relating some numerical invariants of the extension v/v_0. By studying these invariants punctually at t, we show that if there exists a t such that $v/v_0{}^t$ is unique, then $v/v_0{}^{t'}$ is also unique for every t' of minimal deg such that v extends $v_0{}^{t'}$.

The D^h that appears in the above equality is called the henselian defect of v/v_0. The notion of defect, which is central to the ideas of section 2, is introduced in section 1.

We have come to this work from two directions. One is that of the equality $E=IRD^h$, which had been conjectured to hold under certain hypotheses in [12]; the present paper puts [12] in its proper setting and completes the proof of its conjectures. The other direction is that of the uniqueness problem for function fields: an affirmative answer to this problem is given in [7] for v_0 rk 1 complete and is a key step in the proof of the genus reduction inequality of that paper. See also the introduction to [7] for some historical remarks on this problem.

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