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Two torsion and homotopy associative H-spaces

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§ 0. Introduction

In this note we consider the following question:

If Y is a mod 2 H-space, when does $Y \times S^7$ admit the structure of a homotopy associative mod 2 H-space?

There are several examples that are revealing. First, it is well known that the seven-sphere admits the structure of an H-space, but does not admit a homotopy associative structure. In the case of Lie groups, it is known that at the prime 2, Spin(8) is homotopy equivalent to Spin(7) $\times S^7$ and Spin(7) is homotopy equivalent to $G_2 \times S^7$. Among all the compact simply connected simple Lie groups, only G_2, F_4, Spin(7) and Spin(8) have a subHopf algebra over the Steenrod algebra of the following form

$$A = \frac{\mathbf{Z}_2[x]}{x^4} \otimes \wedge (Sq^2 x) = H^*(G_2; \mathbf{Z}_2), \quad \deg x = 3. \tag{0.1}$$

In this paper we show that this is the key factor in determining if a finite H-space producted with a seven-sphere can admit a homotopy associative H-structure. This can be summarized by the following theorems.

Theorem A. *Let Y be a finite 1-connected complex and suppose $H^*(Y; \mathbf{Z}_2)$ does not contain any subalgebras over the Steenrod algebra of type A. Then $Y \times S^7$ cannot be a homotopy associative H-space.*

Theorem B. *Let Y be a finite 1-connected complex and suppose $H^*(Y; \mathbf{Z}_2)$ has at most one subalgebra of type A over the Steenrod algebra. Then $Y \times (S^7)^k$ cannot be a homotopy associative for $k \geq 3$.*

The first results concerning products with S^7 and homotopy associativity were due to Goncalves, [2], who proved that if Y is any simply-connected compact simple Lie group other than G_2 and Spin(7), then $Y \times S^7$ cannot be a homotopy-associative H-space, even when localized at the prime two. Hubbuck [3] showed that the two-torsion is necessary for their products with S^7 to be the homotopy

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