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Construction of irreducible unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$

Dedicated to Professor Nobuhiko Tatsuuma on his sixtieth birthday

By

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Introduction.

For a set I, we denote by $\mathfrak{S}_I$ the group of all finite permutations on I. In this paper, we study irreducible unitary representations (=IURs) of the infinite symmetric group $\mathfrak{S}_N$, denoted also by $\mathfrak{S}_\infty$. We consider it as an infinite discrete group, of non type I, and apply our results in the previous paper [DG] (=[8]), getting a big family of completely new type of IURs.

Representations of the infinite symmetric group have been studied from many standpoints. All the indecomposable positive-definite class functions (or characters) have already been determined by Thoma [21]. They are also studied recently by Vershik and Kerov from different points of view ([9], [22], [23]). When we introduce a certain non-discrete topology in $\mathfrak{S}_\infty$, it becomes of type I and its IURs can be completely determined as shown by Lieberman ([11], [12]). Cf. also O'lshanskii [17] from this point of view. We have also other works ([3], [5], [7] etc.), rather operator algebra theoretic.

Very recently a new type of IURs has been constructed by Obata [16]. Discussions with him on his study and on Saito's [18] are one of our motivations of the present work, and discussions with Hashizume on his work [6] were also inspiring.

In our previous paper [DG], we studied a general theory of representations of infinite discrete groups, and applied it to wreath product groups $\mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A$ of a group T with the permutation group $\mathfrak{S}_A$, where $D_A(T)=\prod'_{\alpha\in A}T_\alpha$, $T_\alpha=T$ $(\alpha\in A)$, is the restricted direct product. We consider a family $\mathfrak{A}(\mathfrak{S}_A(T))$ of subgroups of the form $H=\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma)$, where $A=\coprod_{\gamma\in\Gamma}A_\gamma$ is a partition of A and T_γ's are subgroups of T. Further consider a family $\mathfrak{R}_H$ of IURs of H coming naturally from characters χ_γ of $\mathfrak{S}_{A_\gamma}$, IURs $\rho^\gamma_{T_\gamma}$ of T_γ and reference vectors to form tensor products, and put $\mathfrak{R}(\mathfrak{S}_A(T))=\bigcup_H\mathfrak{R}_H$ $(H\in\mathfrak{A}(\mathfrak{S}_A(T)))$. Then, in case $|T|<\infty$, the induced representations

$$\mathrm{Ind}_H^{\mathfrak{S}_A(T)}\pi,\quad H\in\mathfrak{A}(\mathfrak{S}_A(T)),\quad \pi\in\mathfrak{R}_H\subset\mathfrak{R}(\mathfrak{S}_A(T)),$$

give always IURs of $\mathfrak{S}_A(T)$ if $|\Gamma_f|\leqq 1$ with $\Gamma_f=\{\gamma\in\Gamma\,;\,|A_\gamma|<\infty\}$ and $\mathrm{Ind}_{T_\gamma}^T\rho^\gamma_{T_\gamma}$ is irreducible for $\gamma\in\Gamma_f$. Moreover the equivalence relations among these IURs are also completely determined.

For our study on the infinite symmetric group $G=\mathfrak{S}_N$ in the present paper, we

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