J. Math. Kyoto Univ. (JMKYAZ)
31-2 (1991) 419-466

Logarithmic Enriques surfaces

By

De-Qi ZHANG

Introduction

Normal projective surfaces with only quotient singularities appear in studies of threefolds and semi-stable degenerations of surfaces (cf. Kawamata [5], Miyanishi [6], Tsunoda [11]). We have been interested in such singular surfaces with logarithmic Kodaira dimension $-\infty$ (cf. Miyanishi-Tsunoda [8], Zhang [12, 13]). In the present paper, we shall study the case of logarithmic Kodaira dimension 0.

Let $\bar{V}$ be a normal projective rational surface with only quotient singularities but with no rational double singular points. Let $K_{\bar{V}}$ be the canonical divisor of $\bar{V}$ as a Weil divisor. We call $\bar{V}$ a logarithmic Enriques surface if $H^1(\bar{V}, \mathcal{O}_{\bar{V}})=0$ and $K_{\bar{V}}$ is a trivial Cartier divisor for some positive integer N. The smallest one of such integers N is called the index of $K_{\bar{V}}$ and denoted by Index$(K_{\bar{V}})$ or simply by I. Since $IK_{\bar{V}}$ is trivial, there is a $\boldsymbol{Z}/I\boldsymbol{Z}$-covering $\pi: \bar{U}\to\bar{V}$, which is unique up to isomorphisms and étale outside Sing$\bar{V}$. Then $\bar{U}$, called the canonical covering of $\bar{V}$, is a Gorenstein surface, and the minimal resolution of singularities of $\bar{U}$ is an abelian surface or a $K3$-surface.

Let $f: V\to\bar{V}$ be a minimal resolution of singularities of $\bar{V}$ and set $D:=f^{-1}(\mathrm{Sing}\bar{V})$. We often confuse $\bar{V}$ deliberately with (V, D) or (V, D, f).

§1 is a preparation and contains a proof of an inequality (cf. Proposition 1.6) which plays an important role in the whole theory; in particular, if $I\geqq 3$ then $c:=\#(\mathrm{Sing}\bar{V})\leqq(D, K_V)\leqq c-1-(K_{\bar{V}}^2)$, and it $I\geqq 4$ then $c<-3(K_{\bar{V}}^2)$. In §2, it is proved that if a positive integer p is a factor of I then $\bar{U}/(\boldsymbol{Z}/p\boldsymbol{Z})$ is a logarithmic Enriques surface, as well. We also prove that $I\leqq 66$; this result is originally due to S. Tsunoda. Moreover, $I\leqq 19$ if I is a prime number. §§3-5 are devoted to the proofs of the following three theorems:

Theorem 3.6. *Let $\bar{V}$ or synonymously (V, D) be a logarithmic Enriques surface with* Index$(K_{\bar{V}})=2$. *Then there is a logarithmic Enriques surface $\bar{W}$ or (W, B) with* Index$(K_{\bar{W}})=2$ *and* $\#(\mathrm{Sing}\bar{W})=1$ *such that V is obtained from W by blowing up all singular points of B (i.e., intersection points of irreducible components of B) and then blowing down several (-1)-curves on the blown-up surface.*

Moreover, $\#(\mathrm{Sing}\bar{U})=\#(\mathrm{Sing}\bar{V})\leqq\#\{$*irreducible component of* $D\}\leqq 10$ (*cf. Lemma* 3.1). *The case with* $\#(\mathrm{Sing}\bar{V})=10$ *occurs* (*see Example* 3.2) *and, in this case, there is a* (-2)-*rod of Dynkin type* A_{19} *on* U (*cf. Cor.* 3.10).

Communicated by Prof. Nagata.
Received June 23, 1989