J. Math. Kyoto Univ. (JMKYAZ)
31-2 (1991) 329-358

Birational endomorphisms of the affine plane

By

D. DAIGLE*

Let A^2 be the affine plane over an algebraically closed field k. The birational endomorphisms of A^2 are those maps $A^2 \to A^2$ given by endomorphisms ϕ of the polynomial algebra $k[X, Y]$ such that $k(\phi(X), \phi(Y)) = k(X, Y)$. The simplest non-automorphic example that comes to mind is the ϕ_0 defined by $\phi_0(X) = X$ and $\phi_0(Y) = XY$. The birational endomorphisms of A^2 given by $\phi = u \circ \phi_0 \circ v$, where u and v are any automorphisms of $k[X, Y]$, are called *simple affine contractions in* A^2. The question whether every birational endomorphism of A^2 is a composite of simple affine contractions arose in the early seventies, in Abhyankar's seminar at Purdue University. That question was answered negatively by K.P. Russell who, in conversations with A. Lascu, constructed an irreducible birational endomorphism with three fundamental points. He soon exhibited a whole zoo of irreducible endomorphisms, some of them having infinitely near fundamental points. Its diversity shows that to give a reasonably complete *classification* of all birational endomorphisms of A^2 is likely to be interesting and difficult. The aim of this paper is to make some contributions to that problem.

The methods by which Russell constructed those endomorphisms and proved their irreducibility consist in a detailed analysis of the configuration of missing curves (see (1.2f) for definition). Our approach is essentially an elaboration of Russell's methods, and includes the use of some graph-theoretic techniques (weighted graphs, dual trees). Note that the last section of this paper is in fact an appendix which gathers some definitions and facts in the theory of weighted graphs.

The first three sections study birational morphisms $f: X \to Y$ of nonsingular surfaces, and the fourth section concentrates on the case $X = Y = A^2$. The most interesting results are, we think, those numbered (2.1), (2.9), (2.17), (4.3), (4.4), (4.11), (4.12) and (4.13). We point out that the last section of our paper [2] classifies the irreducible $f: A^2 \to A^2$ with two fundamental points.

Throughout this paper, our ground field is an arbitrary algebraically closed field k, all curves and surfaces are irreducible and reduced, all surfaces are nonsingular and the word "point" means "closed point". The domain (dom (f)) and codomain (codom (f)) of any birational morphism f under consideration will be tacitly assumed to be (nonsingular) surfaces. If X is a surface, Div (X) is its group of divisors and Cl (X) its divisor class group; "X is factorial" means that X is the affine spectrum of a U.F.D.; "X has

* During much of this work the author enjoyed the financial support of NSERC (Canada) and the hospitality of the University of Kentucky.
Communicated by Prof. Nagata.
Received May 16, 1988; Revised April 20, 1990