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Relations on pfaffians I: plethysm formulas

By

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1. Introduction

Let R be a commutative ring with unity, and fix an integer $n \geq 1$. Suppose x_{ij} be variables with $1 \leq i < j \leq n$. Denote by $S = R[X]$ the polynomial ring with $n(n-1)/2$ variables x_{ij}. Put $x_{ij} = -x_{ji}$ for $1 \leq i < j \leq n$ and $x_{ii} = 0$ for $i = 1, \ldots, n$. (x_{ij}) is the generic n by n antisymmetric matrix with entries in S. For a positive integer t such that $1 < 2t \leq n$ and a strictly increasing sequence $(1 \leq) p_1 < \cdots < p_{2t} (\leq n)$,

$$\frac{1}{2^t t!} \sum_{\sigma \in \mathbf{S}_{2t}} (\operatorname{sgn} \sigma) x_{p_{\sigma(1)} p_{\sigma(2)}} x_{p_{\sigma(3)} p_{\sigma(4)}} \cdots x_{p_{\sigma(2t-1)} p_{\sigma(2t)}}$$

is called a *$2t$-order pfaffian.* (This polynomial is defined over an arbitrary commutative ring R.) It is well-known that the square of this $2t$-order pfaffian coincides with the determinant of the $2t$ by $2t$ antisymmetric matrix $(x_{p_i p_j})_{i,j=1,\ldots,2t}$. We denote by Pf_{2t} the ideal generated by all $2t$-order pfaffians of (x_{ij}) and call it the *pfaffian ideal* of order $2t$.

It is well-known that, if R is Gorenstein, Pf_{2t} is a Gorenstein ideal with $\operatorname{grade}(Pf_{2t}) = hd_S(S/Pf_{2t}) = (n-2t+1)(n-2t+2)/2$ ([7] or [9]). Furthermore any Gorenstein subscheme of codimension 3 is known to be defined by certain pfaffians of a certain antisymmetric matrix ([3]).

The main purpose of this article is to investigate when the first syzygy modules of pfaffian ideals are generated by their relations of degree 1. When R contains the rationals $\mathbf{Q}$, any relation on pfaffians can be written by relations of degree 1 (in this case all the syzygies have been determined in [7] or [8]). In the case of arbitrary characteristic, when $t = 1$, $n = 2t$, $n = 2t+1$ ([3]) or $n = 2t+2$ ([14]), minimal free resolutions have been already constructed and the relation modules are generated by relations of degree 1. The main result of this article is

Theorem 5.3. 1. *The first syzygy of the pfaffian ideal Pf_{2t} is generated over $S(\wedge^2 E)$ by relations of degree at most t, i.e.,*

$$\operatorname{Ker}(M_t) = S(\wedge^2 E) \cdot \left(\sum_{r=1}^{t} \operatorname{Ker}(M_{t,r})\right).$$

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