Isogenous tori and the class number formulae

By

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Introduction

T. Ono and J.-M. Shyr generalized Dedekind's class number formulae to a class number formula of an algebraic torus T defined over \mathbf{Q} (cf. [7], [10]). From this generalized class number formula, they obtained a relation between the relative class number of two isogenous tori and their Tamagawa numbers and q-symbols of several maps induced by an isogeny of them (cf. Lemma 1). Here q-symbols of α is defined as follows. Let A, B be commutative groups and α be a homomorphism $A \rightarrow B$. If Ker α and Cok α are both finite, that is, α is admissible, we define the q-symbol of α by putting

$$q(\alpha) = \frac{[\operatorname{Cok} \alpha]}{[\operatorname{Ker} \alpha]},$$

where [X] denotes the order of a finite group X.

Let F be an algebraic number field of finite degree over Q and T be an algebraic torus defined over F. h(T) denotes the class number of T. Consider the following exact sequence of algebraic tori defined over F

$$0 \longrightarrow R_{K/F}^{(1)}(G_m) \longrightarrow R_{K/F}(G_m) \longrightarrow G_m \longrightarrow 0,$$

where K is a finite extension of F and $R_{K/F}$ is the Weil functor of restricting the field of definition from K to F. As a generalization of the formula of Gauss on the genera of binary quadratic forms, T. Ono defined a new arithmetical invariant E(K/F) by putting

$$E(K/F) = h(R_{K/F}(G_m))/(h(R_{K/F}^{(1)}(G_m)) \cdot h(G_m)).$$

In [9], he obtained a formula of E(K/F) expressed in terms of cohomological invariants for K/F. He also defined another invariant E'(K/F), and in [5], we briefly announced similar formula for E'(K/F) when K/F is finite normal. In [6], using I. T. Adamson's non-normal cohomology, we announced that one could generalize these formulae of E(K/F) and E'(K/F) for any finite extension K/F.

In this paper, we shall prove these announced results of [5] and [6] in §1. In §2, we shall show another class number formula for a biquadratic extension K/F. In §3, we shall show this formula implies some class number

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