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Invariant connections for conformal and projective changes

Dedicated to Professor Doctor Makoto Matsumoto on the occasion of his seventieth birthday

By

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In the theory of conformal changes of Riemannian metrics, the Weyl conformal curvature tensor plays an essential role in case of dimension more than three. We consider, however, the theory from another standpoint. In the first section of the present paper we shall define a linear connection on a Riemannian space relative to a given Riemannian space, which is invariant under conformal change of metric. Thus the curvature tensor of the connection is a conformal invariant and the notion of relative conformal flatness is obtained.

The second section is devoted to the theory of projective changes of Finsler spaces in a similar way. Relative to a given Finsler space a projectively invariant nonlinear connection is defined. As a special case we have a Riemannian projective theory which will be developed in the following two sections.

§ 1. Conformal changes of a Riemannian space

Let M be an n-dimensional differential manifold and $T(M)$ its tangent bundle. A coordinate system $x=(x^i)$ in M induces a canonical coordinate system $(x, y)=(x^i, y^i)$ in $T(M)$. We put $\partial_i=\partial/\partial x^i$ and $\dot{\partial}_i=\partial/\partial y^i$.

Let us suppose that there is given on M a Riemannian metric tensor $a_{ij}(x)$. Putting $\alpha=(1/2)a_{ij}y^iy^j$, we denote the Riemannian structure by (M, α). The Christoffel symbols $\{{}_j{}^i{}_k\}$ constructed from a_{ij} are coefficients of the Riemannian connection.

We now consider another arbitrary Riemannian metric tensor $g_{ij}(x)$. Putting $L=(1/2)g_{ij}y^iy^j$, this Riemannina is denoted by (M, L). The Christoffel symbols constructed from g_{ij} are denoted by $\Gamma_j{}^i{}_k$. We put $a=\det(a_{ij})$, $g=\det(g_{ij})$ and

$$C\equiv\frac{1}{n}\log\frac{\sqrt{g}}{\sqrt{a}}, \qquad C_i\equiv\partial_iC. \tag{1}$$

C is a scalar function on M and consequently C_i is covariant vector field on M. Then we have a linear connection

$${}^c\Gamma_j{}^i{}_k\equiv\Gamma_j{}^i{}_k-C_j\delta^i{}_k-C_k\delta^i{}_j+C^ig_{jk}, \tag{2}$$

where $C^i\equiv g^{ij}C_j$. The connection ${}^c\Gamma$ is symmetric but not metrical. We shall call ${}^c\Gamma$ the C-connection relative to α. We have the following important theorem:

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