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Openness of stability

By

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Let X be a smooth projective variety of dimension n. Let $N.S.(X)$ be the Neron-Severi group of divisors on X modulo numerical equivalence. Then $N.S.(X)$ is a finitely generated abelian group which embeds in $V = N.S.(X) \otimes_{\mathbf{Z}} \mathbf{R}$. By a result of Kleiman [1], there is an open cone C in V such that $C \cap N.S.(X)$ consists of all classes of ample divisors.

Let θ be an ample divisor on X. If $\mathscr{F}$ is a coherent sheaf on X, then $\deg_\theta \mathscr{F} \equiv$ the intersection number $c_1(\mathscr{F}) \cdot \theta^{n-1}$ where $c_1(\mathscr{F})$ is the first Chern class and the slope $\mu_\theta \mathscr{F} \equiv \deg_\theta \mathscr{F} / \text{rank}\, \mathscr{F}$ if $\mathscr{F}$ is not torsion. A vector bundle $\mathscr{W}$ on X is θ-stable if $\mu_\theta(\mathscr{F}) < \mu_\theta(\mathscr{W})$ for all coherent $0 \subsetneq \mathscr{F} \subsetneq \mathscr{W}$.

In this paper we propose to prove

Theorem 1. *There is an open cone $D(\mathscr{W}) \subset C$ such that $N.S.(X) \cap D(\mathscr{W})$ consists of the classes of θ such that $\mathscr{W}$ is θ-stable.*

This result may be proven analytic over $\mathbf{C}$ replacing $N.S.(X)$ by the real $H^{1,1}$-classes and C by the classes of Kähler metrices. In this case the result follows from the openness of the differential operator in the equation for a Hermitian-Einstein metric on stable bundles using the Donaldson-Uhlenbeck-Yau theorem. Thus our result is mostly interesting in characteristic p unless one just wants an algebraic proof.

Also there are the openness theorems of Maruyama [2] where the polarization is essentially fixed but $\mathscr{W}$ and X vary algebraically. There should be a common generalization of our results but this would be too complicated in seeing the ideas clearly.

§1. Testing for stability

We first note

Lemma 2. *The θ-stability of $\mathscr{W}$ is equivalent to the condition*

*) *for all $0 \leq i < \text{rank}\, \mathscr{W}$ and all invertible sheaves $\mathscr{L}$ on X such that $\deg_\theta \mathscr{L} \geq i\mu_\theta(\mathscr{W})$, there is no non-zero section of $\Lambda^i \mathscr{W} \otimes \mathscr{L}^{\otimes -1}$ that satisfies the Plücker relations at a generic point of X.*

Proof. If $0 \subset \mathscr{F} \subset \mathscr{W}$ is a destabilizing $\mathscr{F}$ then $\Lambda^i \mathscr{W} \otimes (\Lambda^i \mathscr{F})^{dual}$ has a non-zero section where $i = \text{rank}\, \mathscr{F}$. Now $(\Lambda^i \mathscr{F})^{dual\ dual} = \mathscr{L}$ is invertible and

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