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On mod p cohomology of the space X_Γ and mod p trace formula for Hecke operators

By

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1. Introduction

Let $M_2\mathbf{Z}$ denote the set of 2×2 matricies over $\mathbf{Z}$. The semigroup $M_2\mathbf{Z}$ acts on the 2-torus T^2 as endomorphisms in the standard way. For any subgroup Γ of $M_2\mathbf{Z}$, we can define the semi-direct product $\Gamma \ltimes T^2$. We denote the classifying space $B(\Gamma \ltimes T^2)$ by X_Γ. Our main interest is in the case of $\Gamma = SL_2\mathbf{Z}$ or congruent modular groups. In this case it is known [3] that there is an isomorphism

$$H^{2n+1}(X_\Gamma; \mathbf{R}) \cong H^1(\Gamma; H^{2n}(BT^2; \mathbf{R})),$$

where the right hand side is the cohomology of the group Γ with the coefficient module $H^{2n}(BT^2; \mathbf{R})$. Then the Eichler-Shimura homomorphism turns out to be a homomorphism

$$\phi: M_k \to H^{2k-3}(X_\Gamma; \mathbf{R}),$$

where M_k is the space of modular forms of weight k for Γ. Moreover it is known [6] that we can define a Hecke operator $T(n)$ on $H^*(X_\Gamma; M)$ for any module M induced from a stable self map of X_Γ, and ϕ commutes with Hecke operators. Now the theorem of Eichler and Shimura [7] asserts that $\phi: S_k \cong H_P^1(\Gamma; H^{2k-4}(BT^2; \mathbf{R}))$, where S_k is the space of cusp forms of weight k and H_P^1 means the parabolic cohomology of Γ. Then it is also known [3] that

$$\dim_{\mathbf{R}} H^{2k-3}(X_\Gamma; \mathbf{R}) = \dim_{\mathbf{R}} H_P^1(\Gamma; H^{2k-4}(BT^2; \mathbf{R})) + \nu,$$

where ν is the number of equivalent classes of cusps.

In this note we study $H^*(X_\Gamma; \mathbf{F}_p)$, which has richer structure; the action of Steenrod algebra and the module structure over $H^{ev}(X_\Gamma; \mathbf{F}_p)$. We shall show that $H^{ev}(X_\Gamma; \mathbf{F}_p) \cong \mathbf{F}_p[x, y]^\Gamma$, the ring of Γ-invariants. In the case of $\Gamma = SL_2\mathbf{Z}$, $\mathbf{F}_p[x, y]^\Gamma \cong \mathbf{F}_p[q, q_1]$, $\deg q = p + 1$ and $\deg q_1 = p(p-1)$. It is shown (Corollary 3.3) that $H^*(X_\Gamma; \mathbf{F}_p)/qH^*(X_\Gamma; \mathbf{F}_p)$ satisfies certain periodicity. Using this the $\mathbf{F}_p[q, q_1]$-module structure of $H^*(X_\Gamma; \mathbf{F}_p)$ is determined (Theorem 5.3). As a corollary of those results, we give a formula on mod p value of the trace of Hecke operators acting on cusp forms.

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