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Elliptic 3-folds and Non-Kähler 3-folds

By

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§ 0. Introduction

The purpose of this paper is to study the relationship between Calabi-Yau 3-folds with elliptic fibrations and compact non-Kähler 3-folds with $K=0$, $b_2=0$, $q=0$. The non-Kähler 3-folds referred to here have firstly appeared in Friedman's paper [3]. In this paper he has shown that if there are sufficiently many (mutually disjoint) $(-1, -1)$-curves on a Calabi-Yau 3-fold, then one can contract these curves and can deform the resulting variety to a smooth non-Kähler 3-fold with $K_2=0$, $b_2=0$, $q=0$. For example, in the case of a (general) quintic hypersurface in $\boldsymbol{P}^4$, one can do this procedure for two lines on it. This phenomenon is analogous to the one for (-2)-curves on a $K3$ surface. In fact a (-2)-curve on a $K3$ surface often disappears in a deformation and this fact just says that one can contract this (-2)-curve to a point and can deform the resulting variety to a (smooth) $K3$ surface. By this phenomenon, we can explain the varience of the Picard numbers of $K3$ surfaces in deformations and it is well-known that a general point of the moduli space of $K3$ surfaces corresponds to a non-projective (but Kähler) $K3$ surface on which there are no (-2)-curves. Taking such a non-projective surface into consideration, one has a famous theorem that two arbitrary $K3$ surfaces are connected by deformations. There is, however, a difference between Calabi-Yau 3-folds and $K3$ surfaces, that is, a $(-1, -1)$-curve never disappears like a (-2)-curve in deformations. This is closely related to the fact that Calabi-Yau 3-folds have a large repertory of topological Euler numbers. For the speculation around this area, one may refer to M. Reid's paper [12].

The main result of this paper is the following:

Theorem A. *Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then the bimeromorphic class of X is obtained as a semi-stable degeneration of a compact non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$, i.e. there is a surjective proper map f of a smooth 4-dimensional variety $\mathcal{X}$ to a 1-dimensional disc Δ such that*

1) *$f^{-1}(t)$ is a compact non-Kähler 3-fold with $K=0$, $b_2=0$, $q=0$ for $t\in\Delta^*$,*
2) *$f^{-1}(0)=\sum_{i=0}^{n} X_i$ is a normal crossing divisor of $\mathcal{X}$, and*
3) *X_0 is bimeromorphic to X and other X_i's are in the class $\mathcal{C}$.*

Here we will explain the motivation of the formulation in Theorem A. If there are

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