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On the structure of Cousin complexes

By

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0. Introduction

Throughout the paper, A will denote a commutative Noetherian ring (with non-zero identity), and M will denote an A-module. It should be noted that M need not be finitely generated. The Cousin complex $C(M)$ for M is described in [3, Section 2]: it is a complex of A-modules and A-homomorphisms

$$0 \xrightarrow{b^{-2}} M \xrightarrow{b^{-1}} B^0 \xrightarrow{b^0} B^1 \to \cdots \to B^n \xrightarrow{b^n} B^{n+1} \to \cdots$$

with the property that, for each $n \in \mathbf{N}_0$ (we use $\mathbf{N}_0$ to denote the set of non-negative integers),

$$B^n = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \mathfrak{p} = n}} (\mathrm{Coker}\, b^{n-2})_{\mathfrak{p}} .$$

(Here, for $\mathfrak{p} \in \mathrm{Supp}(M)$, the notation $\mathrm{ht}_M \mathfrak{p}$ denotes the *M-height of* $\mathfrak{p}$, that is the dimension of the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$; the dimension of a non-zero module is the supremum of lengths of chains of prime ideals in its support if this supremum exists and ∞ otherwise.)

Cohen-Macaulay modules can be characterized in terms of the Cousin complex: a non-zero finitely generated A-module N is Cohen-Macaulay if and only if $C(N)$ is exact [4, (2.4)]. Also, the Cousin complex provides a natural minimal injective resolution for a Gorenstein ring: see [3, (5.4)].

The Cousin complex $C(M)$ will play a major rôle in this paper. It is a special case of a more general complex which can be constructed whenever we have a filtration $\mathscr{F}$ of $\mathrm{Spec}(A)$ which admits M [8, 1.1]; this more general complex is called the Cousin complex for M with respect to $\mathscr{F}$ and is denoted by $C(\mathscr{F}, M)$. As this complex will also feature prominently in this paper, it is appropriate for us to recall the details of its construction and definition from [8, Section 1].

A *filtration* of $\mathrm{Spec}(A)$ is a descending sequence $\mathscr{F} = (F_i)_{i \in \mathbf{N}_0}$ of subsets of $\mathrm{Spec}(A)$, so that

$$\mathrm{Spec}(A) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots ,$$

with the property that, for each $i \in \mathbf{N}_0$, each member of $\partial F_i = F_i \backslash F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathscr{F}$ *admits* M if $\mathrm{Supp}(M) \subseteq F_0$.

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