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Remarks on torus principal bundles

By

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In this paper we study principal bundles $X \xrightarrow{\pi} M$ over a compact complex manifold M whose structure group is a compact complex torus $T = V/\Lambda$. The total space X of such a principal bundle is usually not a Kähler space even if the base manifold M is.

Typical examples are Hopf manifolds, or the Calabi-Eckmann manifolds diffeomorphic to a product of spheres. These are principal bundles over a product of projective spaces, the fibre is an elliptic curve. Those and other special examples have been studied in detail, see [Cal-Eck], [Maeda], [Nakamura], [Akao].

We develop the theory starting from the base manifold M, often assuming that it (i.e. $\mathrm{H}^2(M)$) has a Hodge decomposition. For a T-principal bundle $X \xrightarrow{\pi} M$ we define a characteristic class $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \Lambda)$ (1.3) and invariants $\varepsilon\colon \mathrm{H}_T^{0,1} \to \mathrm{H}_M^{0,2}$, $\gamma\colon \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{1,1}$ (1.5). It will turn out that these can be computed from $c^{\mathbf{Z}}$ and determine the d_2 differentials of the Leray spectral sequence converging to $\mathrm{H}^{\cdot}(X, \mathbf{C})$ and of a spectral sequence converging to $\mathrm{H}_X^{\cdot\cdot}$ (with a variant computing $\mathrm{H}^{\cdot}(\Theta_X)$). This spectral sequence was constructed by Borel in his appendix to [Hirzebruch] and was used there to compute the Hodge ring of Calabi-Eckmann manifolds. Since in our case all those spectral sequences degenerate on E_3-level, Betti numbers, Hodge numbers, and the space of infinitesimal deformations of X can be computed in general (Theorem 1.6).

In bundles with $\varepsilon = 0$ the torus T can be replaced by any other torus of the same dimension (e.g. Calabi-Eckmann manifolds), whereas for $\varepsilon \neq 0$ (e.g. Iwasawa manifold) the periods of T must be related to intrinsic data of M (Chapter 7, Chapter 8).

If M is simply-connected, then it is fairly easy to construct simply-connected bundles, even with first Chern class $c_1(X) = 0$. They do not carry a Kähler metric by Blanchard's theorem (1.7), in fact they cannot carry a complex Kähler structure for purely topological reasons (11.4).

If moreover M is a complex surface and T an elliptic curve, then we get a lot of interesting simply-connected complex threefolds with $c_1 = 0$. According to Wall's classification of real six-dimensional manifolds, the only diffeomor-

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