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Range characterization of Radon transforms on S^n and P^nR

By

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0. Introduction

It is one of the most important problems in integral geometry to characterize the ranges of Radon transforms. F. John [9] gave the first answer to this problem. His result is that the range of the X-ray transform on $\mathbf{R}^3$ is characterized by a second order ultrahyperbolic differential operator. Gelfand, Graev, and Gindikin [1] extended John's result; they characterized the ranges of d-plane Radon transforms on $\mathbf{R}^n$ and $\mathbf{C}^n$ by a system of second order differential operators on an affine Grassmann manifold. Farthermore, Gonzalez [4] gave a simple characterization of it by an invariant differential operator on an affine Grassmann manifold. Grinberg [5] characterized the range of the projective k-plane Radon transform on the n-dimensional real projective space $\mathbf{P}^n\mathbf{R}$ and the n-dimensional complex projective space $\mathbf{P}^n\mathbf{C}$ by a system of second order differential operators, and in [10], we gave another type of range characterization for the Radon transform on a complex projective space; we characterized the range by a single differential operator which is a fourth order invariant differential operator on a complex Grassmann manifold and which is ultrahyperbolic type of differential operator.

In this paper, we examine mainly the range of the Radon transform $R = R_l$ on the n-dimensional sphere $\mathbf{S}^n$ for $1 \leq l \leq n-2$, which we define by integrating a function f on $\mathbf{S}^n$ over an oriented l-dimensional totally geodesic sphere ξ, that is, we define R as follows

$$Rf(\xi) = \frac{1}{\mathrm{Vol}(\mathbf{S}^l)} \int_{x \in \xi} f(x)\, dv_\xi(x),$$

where $dv_\xi(x)$ is the canonical measure on $\xi \subset \mathbf{S}^n$. This Radon transform R maps smooth functions on $\mathbf{S}^n$ to smooth functions on $\widetilde{Gr}_{l+1,n+1}$, the compact oriented real Grassmann manifold, that is, $R: C^\infty(\mathbf{S}^n) \to C^\infty(\widetilde{Gr}_{l+1,n+1})$.

The main result of this paper is the following:

Theorem. *There exists a fourth order invariant differential operator P on $\widetilde{Gr}_{l+1,n+1}$ such that the range* $\operatorname{Im} R$ *of R is identical with its kernel* $\operatorname{Ker} P$, *i.e.*,

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