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Quasi sure quadratic variation of smooth martingales

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1. Introduction

Suppose that $\boldsymbol{M}=\{M_t, t\in I\}$ is a continuous L^p-martingale ($p\geq 2$), where I is an interval of $\boldsymbol{R}_+=[0,\infty)$ (may be $\boldsymbol{R}_+$ itself). By the well-known Doob-Meyer decomposition theorem, there exists a unique increasing process $\langle \boldsymbol{M}\rangle =\{\langle M\rangle_t, t\in I\}$ such that $\boldsymbol{M}^2-\langle \boldsymbol{M}\rangle$ is a continuous $L^{p/2}$-martingale. Moreover, P. W. Millar [12] and D. Nualart [13] showed that the process $\langle \boldsymbol{M}\rangle$ can be obtained as the $L^{p/2}$-limit of sums of the form $\sum M(\Delta_i)$, where $\{\Delta_i\}$ is a subdivision of the interval I, as $\max_i|\Delta_i|\to 0$.

In the present paper we propose to study the quasi sure properties of the quadratic variation of *smooth martingales*, a notion introduced recently by P. Malliavin and D. Nualart [9]. We shall prove that the process of the quadratic variation of a smooth martingale admits an ∞-modification, which can be constructed as the quasi sure limit of sums of the form $\sum M(\Delta_i)$. Our tool is the quasi sure version of Kolmogorov's criterion for the continuity of trajectories of stochastic processes (cf. [17]). Necessary estimations which enable us to apply this criterion will be obtained. This makes the subject of section 3. In section 4 we will be able to extend the results of section 3 to the case of two-parameter smooth martingales. At last in section 5 we discuss possible extensions and applications. We prove, in particular, that the quadratic variation of the Brownian motion is quasi surely t.

The main results of this paper were announced in [21].

2. Preliminaries

Now let us recall and fix some notations and notions. We shall work on the probability space (X, H, μ), where X is the space of continuous maps from $[0,1]$ to $\boldsymbol{R}^d$, null at zero; H is the usual Cameron-Martin subspace and μ the standard Wiener measure. Denote by W_{2r}^p the Sobolev space of order $2r$ and of power p over X and W_∞ their intersection over indexes $p>1$ and $r\geq 0$. For any natural number r, two equivalent norms in W_{2r}^p are defined

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