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On hyperplane sections of reduced irreducible varieties of low codimension

By

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1. Introduction

Let X be an arithmetically Cohen-Macaulay variety (subscheme) of codimension 2 in $\boldsymbol{P}^n=\boldsymbol{P}^n(k)$, where k is an algebraically closed field. Let $I=I(X)$ denote the defining ideal of X in the polynomial ring $R=k[x_0, \cdots, x_n]$. By the Hilbert-Burch theorem we may assume that I is minimally generated by the maximal minors of an $r-1$ by r matrix (g_{ij}) of homogeneous elements of R. Let $a_1, \cdots, a_r$ be the degree of these generators. Then I has a minimal free resolution of the form

$$0 \longrightarrow \bigoplus_{i=1}^{r-1} R(-b_i) \xrightarrow{(g_{ij})} \bigoplus_{j=1}^{r} R(-a_j) \longrightarrow I \longrightarrow 0\,,$$

where $b_1, \cdots, b_{r-1}$ are positive integers with $\sum b_i=\sum a_j$. Put $u_{ij}=b_i-a_j$. We have $\deg g_{ij}=u_{ij}$, if $u_{ij}>0$ and $g_{ij}=0$ if $u_{ij}\leq 0$. Under the assumptions $a_1\leq\cdots\leq a_r$ and $b_1\leq\cdots\leq b_{r-1}$, the matrix (u_{ij}) is uniquely determined by X, and it carries all the numerical data about X. One calls (u_{ij}) the *degree matrix* of X [5].

In [24] Sauer proved that an arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ is smoothable if and only if $u_{ii+2}\geq 0$ for $i=1, \cdots, r-2$. At a first glance Sauer's result is surprising in so far as smoothability should solely depend on the Hilbert function of the curve (which of course is determined by the degree matrix but not vice versa). However, as observed by Geramita and Migliore [13], this numerical condition of the degree matrix can indeed be expressed in terms of the Hilbert function of C.

On the other hand, as noted in [13], Sauer ([24]) proved, though not explicitly stated, that a matrix of integers $u_{ij}=b_i-a_i$, where $a_1\leq\cdots\leq a_r$ and $b_1\leq\cdots\leq b_{r-1}$ are two sequences of positive integers with $\sum a_i=\sum b_j$, is the degree matrix of a smooth arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ if and only if $u_{ii+2}>0$ for $i=1, \cdots, r-2$. Here the reference to the stronger numerical invariant, the degree matrix, is indispensible, since the Hilbert function

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