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On Mordell-Weil lattices of higher genus fibrations on rational surfaces

By

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§ 0. Introduction

(0.1) Let $f: X \to C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C defined over an algebraically closed field k of characteristic zero, and let K be the rational function field of C. We assume that there exists a section O of f. For such a fibration, we can define the Mordell-Weil group to be the group of the K-rational points of the Jacobian J_Γ of the generic fiber Γ/K of f. Under the suitable condition, the Mordell-Weil group $J_\Gamma(K)$ is a finitely generated abelian group, so we define the Mordell-Weil rank r to be the rank of its free part. In this note we first prove the following theorem which gives an upper bound of the Mordell-Weil rank r for fibrations of genus g on rational surfaces X.

Theorem A (*cf. Theorem* 2.8). *Let X be a smooth rational surface with a relatively minimal fibration $f: X \to \boldsymbol{P}^1$ of curves of genus $g \geq 1$. Then we have*

$$r = \mathit{rank}\ J_\Gamma(K) \leq 4g+4.$$

Moreover we have the equality $r=4g+4$ if and only if $f: X \to \boldsymbol{P}^1$ is a hyperelliptic fibration with $K^2_{X/\boldsymbol{P}^1}=4g-4$ such that all fibers of f are irreducible. Here $K_{X/\boldsymbol{P}^1}=K_X \otimes f^*(K^{-1}_{\boldsymbol{P}^1})$ denotes the relative canonical bundle of f.

(0.2) If $f: X \to \boldsymbol{P}^1$ is a relatively minimal rational elliptic surface with a section, it can be obtained as the minimal resolution of its Weierstrass model, and it is easy to see that all fibers of f are irreducible if and only if its Weierstrass model is smooth. Moreover we can easily construct a smooth Weierstrass fibration $f: X \to \boldsymbol{P}^1$ such that X is a rational surface. The Mordell-Weil rank of such a fibration is maximal ($=8$) because we always have $K^2_{X/\boldsymbol{P}^1}=0$ from the theory of elliptic surfaces due to Kodaira [Kod].

When $g \geq 2$, we can also give a series of examples of rational surfaces X with fibrations of curves of genus g whose Mordell-Weil rank is maximal,

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