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On Beauville's conjecture and related topics

By

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The main purpose of this paper is to discuss a fifteen years old conjecture proposed by A. Beauville on the number of singular fibres of a semi-stable fibration over $\mathbf{P}^1$ ([B1]). The departure point is the so-called function field analog of second Shafarevich's conjecture. Precisely, let $f: X \to C$ be a non-isotrivial fibration over a complex algebraic curve C whose generic fibre is a smooth projective irreducible curve F of genus $g \geq 1$. Put

$$s = \text{the number of } S,\ S = \{t \in C: X_t = f^{-1}(t) \text{ is singular}\}.$$

Shafarevich's conjecture (the function field case). $s > 0$ if $C \simeq \mathbf{P}^1$.

I. Shafarevich proved this statement in [Sh]. By using the action of automorphism group on $\mathbf{P}^1$ A. Parshin ([Par1]) has established that $s \geq 3$ (see also [B1]). Note that in any characteristic (but with a semi-stability condition) the same result was obtained by L. Szpiro ([Sz]). In fact in the semi-stable case over $\mathbf{C}$ a more precise bound was given by A. Beauville ([B1]). Let $g(\tilde{X}_t)$ denote the genus of the normalization of X_t, ρ_2 "the number of transcendental cycles" of X. Let r be the defect relating the Picard number ρ of X and numbers of components of singular fibres (see A.2.2, Appendix A). There is a necessary and sufficient condition for s to be 4 ([B1], cf. also Appendix A).

Theorem (A. Beauville). Let $f: X \to \mathbf{P}^1$ be a semi-stable non-isotrivial fibration. Assume that $g \geq 1$ then $s \geq 4$.

Moreover $s = 4$ if and only if the folowing conditions hold

1) $\rho_2 = 0$,
2) $g(\tilde{X}_t) = g_0$ $\forall t \in S$, where $g_0 = \dim$ of the fixed part of $\mathrm{Pic}^0(X/\mathbf{P}^1)$,
3) $r = 0$,
4) $g_0 = 0$.

Furthermore A. Beauville (*loc. cit.*) constructed some examples with $s = 4$: all those fibrations are elliptic (see also [B2], where he has given a complete classification of all such elliptic fibrations - six cases). In fact A. Beauville was tending to conjecture the following

Beauville's conjecture ([B1]). $s \geq 5$ if $g > 1$.

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