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Relations on Pfaffians: number of generators

Dedicated to Mrs. Nishikawa

By

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Introduction

Let R be a commutative ring with unity, and consider a polynomial ring $S=R[x_{ij}]_{1\leq i<j\leq n}$, where n is a positive integer. With letting $x_{ji}=-x_{ij}$ and $x_{ii}=0$, we can form a generic alternating matrix (x_{ij}). The ideal Pf_{2t} of S generated by all $2t$-subPfaffians of (x_{ij}) is called the *(generic) Pfaffian ideal* of order $2t$.

The main purpose of this article is to determine the number of minimal generators of the relation module, or the fisrt syzygy module of a Pfaffian ideal provided R is a field. In the following cases, the relation module of the Pfaffian ideal is known to be generated by linear relations (i.e., elements of degree $t+1$, because $2t$-Pfaffians are homogeneous of degree t): the case $R\supset\mathbf{Q}$ [10, 12], $t=1$, $n=2t$ (trivial), $n=2t+1$ [5] or $n=2t+2$ [21]. Moreover, Kurano [16] proved that when R is a field of characteristic $p>0$ and if $2p>n-2t$, then the first syzygy of the Pfaffian ideal Pf_{2t} is generated by the linear relations.

On the other hand, Kurano showed that when $n=8$ and $t=2$, we need a new generator of degree $t+2$ of the first syzygy when R is a field of characteristic two [17]. In particular, we see that there is no minimal free resolution of generic Pfaffian ideals over the ring of integers $\mathbf{Z}$ in general.

Our main result is

Theorem 7.8. *Let K be a field of characteristic p.*

1 *If $p\neq 2$, then the first syzygy module of Pf_{2t} as an S-module is generated by linear relations. Or equivalently, we have $\beta^{p}_{2,j}=0$ for $j\neq t+1$.*

2 *If $p=2$, then we have*

$$\mathrm{Tor}_2^S(S/Pf_{2t}, K)\cong[\mathrm{Tor}_2^S(S/Pf_{2t}, K)]_{t+1}\oplus\bigoplus_{i=1}^{[\log_2 t]}\wedge^{2(t+2^i)}F$$

as $\mathrm{GL}(F)$-modules so that $\beta^2_{2,j}=0$ unless $j=t+2^i$ for some $0\leq i\leq[\log_2 t]$.

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