J. Math. Kyoto Univ. (JMKYAZ)
35-3 (1995) 357-376

Two-dimensional Finsler spaces whose geodesics constitute a family of special conic sections

By

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To find the Finsler spaces having a given family of curves as the geodesics is an interesting problem for geometricians and will be an important problem from the standpoint of applications [1]. A previous paper [5] of the present author gave the complete solutions of this problem in the two-dimensional case, and another paper [7] may be regarded as the first continuation of [5]. The present paper is the second continuation.

The purpose of the present paper is to give a geometrical development of the preceding two papers above. The given families of curves treated in this paper consist of semicircles, parabolas and hyperbolas respectively on the semiplane. Some preliminaries are necessary and, in particular, the formula (1.20) for the functions $G^i(x, y)$ will enable us to obtain them easily without finding the fundamental tensor.

§1. Preliminaries

Let us consider an n-dimensional Finsler space $F^n=(M^n, L(x, y))$ on an underlying smooth manifold M^n with the fundamental function $L(x, y)$. The fundamental tensor $g_{ij}(x, y)$, the angular metric tensor $h_{ij}(x, y)$ and the normalized supporting element $l_i(x, y)$ of F^n are defined respectively by

$$g_{ij}=h_{ij}+l_il_j, \quad h_{ij}=LL_{(i)(j)}, \quad l_i=L_{(i)},$$

where $L_{(i)}=\partial L/\partial y^i$ and $L_{(i)(j)}=\partial L_{(i)}/\partial y^j$.

The geodesic, the extremal of the length integral $s=\int_{t_0}^{t} L(x, y)\,\mathrm{d}t$, $t\geqq t_0$, $y^i=\dot{x}^i=\mathrm{d}x^i/\mathrm{d}t$, along a curve $x^i=x^i(t)$, is given by the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{(i)}-L_i=0, \tag{1.1}$$

where $L_i=\partial L/\partial x^i$ and $y^i=\dot{x}^i$. In terms of $F(x, y)=L^2(x, y)/2$, (1.1) is written in the well-known form

$$\ddot{x}^i+2G^i(x, \dot{x})=h(t)\dot{x}^i, \tag{1.2}$$

where we put

Received in February 7, 1994