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On calculations of zeros of various L-functions

By

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Introduction

As we have shown several years ago [Y2], zeros of $L(s, \Delta)$ and $L^{(2)}(s, \Delta)$ can be calculated quite efficiently by a certain experimental method. Here Δ denotes the cusp form of weight 12 with respect to $SL(2, \mathbf{Z})$ and $L(s, \Delta)$ (resp. $L^{(2)}(s, \Delta)$) denotes the standard (resp. symmetric square) L-function attached to Δ. The purpose of this paper is to show that this method can be applied to a wide class of L-functions so that we can obtain precise numerical values of their zeros.[1]

We organize this paper as follows. In §1, we shall describe basic features of our method of calculation, which is repeated applications of partial summation. In §2, we shall study the r-th symmetric power L-function $L^{(r)}(s, \Delta)$ attached to Δ. Since the cases $r = 1, 2$ are discussed in [Y2], we shall exclusively treat the cases $r = 3, 4$. In §3, we shall study the L-functions attached to modular forms of half integral weight. These L-functions do not have Euler products. Naturally the Riemann hypothesis fails for them; we shall find many zeros off the critical line, though major part of zeros lie on the critical line. We shall also calculate the location of these zeros off the critical line. Though there is some hope to find relations among zeros of L-functions of two modular forms which are in the Shimura correspondence, no explicit results came out so far.

In §4, we shall study L-functions attached to Hecke characters of non-A_0 type of real quadratic fields. D.A. Hejhal showed great interest to make experiments in this case, since coefficients are non-computable combinatorially; hence there is a slight possibility that the Riemann hypothesis may break down for these L-functions. We have made experiments on 44 cases summarized in Table 4.3; so far no counterexamples are found.

In §5, we shall study the Artin L-function attached to a 4-dimensional non-monomial representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. In §6, we shall discuss the controle of error estimates in our calculation. In §7, we shall consider the explicit formula for the L-function attached to a modular form of weight 8 with respect to $\Gamma_0(2)$. We shall compare both sides of the explicit formula numerically. In §8,

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[1] After the publication of [Y2], H. Ishii [Is] published a table of zeros of standard L-functions attached to modular forms for 15 cases. It also comes to the author's notice that a program of the calculation of zeros of $L(s, \Delta)$ is included in "Mathematica" package, following the method of [Y2].

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