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The homotopy type of the space of rational functions

Dedicated Professor Seiya Sasao on his 60th birthday

By

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1. Introduction

For each positive integer d, let Hol_d denote the space of all holomorphic (equivalently, algebraic) maps of degree d from the Riemann sphere $S^2 = \mathbf{C} \cup \infty$ to itself. This space is of interest both from a classical and a modern point of view (see [1], [5]). Let Hol_d^* be the subspace of Hol_d consisting of maps which preserve a basepoint of S^2. It is well known that Hol_1 is the group of fractional linear transformations $PSL_2(\mathbf{C})$ and that Hol_1^* may be identified with the affine transformation group of $\mathbf{C}$. It is an elementary fact that Hol_d and Hol_d^* are connected spaces. The fundamental groups of these spaces are $\mathbf{Z}/2d$, $\mathbf{Z}$ respectively; these computations are due to Epshtein ([6]) and Jones (see [8]). The following more general result was obtained by Segal:

Theorem 0 ([8]). *Let* Map_d *be the space of all continuous maps of degree* d *from* S^2 *to itself and let* Map_d^* *be the subspace consisting of maps* f *such that* $f(\infty) = 1$. *Then the natural inclusion maps induce the following isomorphisms of homotopy groups*:

(1) *If* $k < d$, *then* $\pi_k(\mathrm{Hol}_d^*) = \pi_k(\mathrm{Map}_d^*) = \pi_{k+2}(S^2)$.

(2) *If* $k < d$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Map}_d)$.

The stable homotopy type of Hol_d^* was studied in [3]. In this note we shall extend the above results by determining some further homotopy groups of the space Hol_d. Our results are as follows:

Theorem 1. (1) *For* $k \geq 2$,

$$\pi_k(\mathrm{Hol}_d) = \begin{cases} \pi_k(S^3) & d = 1 \\ \pi_k(S^3) \oplus \pi_k(S^2) & d = 2 \\ \mathbf{Z}/2 & d \geq 3,\ k = 2 \end{cases}$$

(2) *If* $k \geq 3$ *and* $d \geq 3$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(S^3)$.

(3) *In particular, if* $d > k \geq 3$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.

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