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Variation formulas for harmonic modules of domains in $\mathbf{R}^3$

Dedicated to Professor Yukio Kusunoki on his 70th birthday

By

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1. Introduction

Let D be a domain spread over the complex plane $\mathbf{C}$ with C^ω smooth boundary ∂D. Suppose that D has a nono-trivial cycle γ. Then there exists a unique L^2 harmonic differential σ on D such that $\int_\gamma \omega = (\omega, *\sigma)_D$ for all C^∞ closed differentials ω on $\bar{D}$. We put $\mu = \| \sigma \|_D^2$. Then $*\sigma$ and μ are called *the reproducing differential* and *the harmonic module for* (D, γ) (see L. V. Ahofors [2]). The geometric meaning of μ was originally studied by Y. Kusunoki [6] and R. Accola [1]. We now let the domain $D(t)$ over $\mathbf{C}$ and the cycle $\gamma(t) \subset D(t)$ vary C^ω smoothly with a complex parameter t in a disk $B = \{|t| < r\}$, where $D(0) = D$ and $\gamma(0) = \gamma$. For any $t \in B$, we have the reproducing differential $*\sigma(t, z)$ and the harmonic module $\mu(t)$ for $(D(t), \gamma(t))$, so $\mu(t)$ is a function on B. We put $\omega(t, z) = \sigma(t, z) + i * \sigma(t, z) = f(z)\, dz$, $\| \omega \|(t, z) = |f(t, z)|$, and $\frac{\partial \omega}{\partial \bar{t}} = \frac{\partial f}{\partial \bar{t}} dz$ for $z \in D(t)$. We here put $\mathfrak{D} = \cup_{t \in B}(t, D(t))$ and $\partial \mathfrak{D} = \cup_{t \in B}(t, \partial D(t))$. Thus $\mathfrak{D}$ is a complex 2 dimensional domain spread over $B \times \mathbf{C}$. Let $\varphi(t, z)$ be a defining function of $\partial \mathfrak{D}$, that is, $\varphi(t, z)$ is a C^ω function in a neighborhood $\mathcal{V}$ of $\partial \mathfrak{D}$ over $B \times \mathbf{C}$ such that $\mathfrak{D} \cap \mathcal{V}$ (resp. $\partial \mathfrak{D}$) $= \{\varphi < 0$ (resp. $= 0)\}$ and $\frac{\partial \varphi}{\partial z} \neq 0$ on $\partial \mathfrak{D}$. We define, for $(t, z) \in \partial \mathfrak{D}$,

$$k_1(t, z) = \frac{\partial \varphi}{\partial t} \Big/ \left| \frac{\partial \varphi}{\partial z} \right|$$

$$k_2(t, z) = \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\Re \left\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial \varphi}{\partial z \partial \bar{z}} \right\} \Big/ \left| \frac{\partial \varphi}{\partial z} \right|^3. \tag{1.1}$$

Note that neither $k_1(t, z)$ nor $k_2(t, z)$ on $\partial \mathfrak{D}$ depends on the choice of $\varphi(t, z)$. In [4] we call $k_2(t, z)$ *the Levi curvature of* $\partial \mathfrak{D}$ *at* (t, z), and proved the following variation formulas:

$$\frac{\partial \mu(t)}{\partial t} = \frac{1}{2} \int_{\partial D(t)} k_1(t, z) \| \omega \|^2(t, z) |dz|$$

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