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Singular principal normality in the Cauchy problem

By

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1. Introduction

It is well known now, after the work of Hörmander [H] and Alinhac [A], that the two relevant concepts for the uniqueness in the Cauchy problem are principal normality and pseudo-convexity. However there are simple examples of operators for which uniqueness holds but principal normality fails on the initial surface. The purpose of this work, whose starting point has been the understanding of the recent example given in [CD], is to show that, in some cases, one can relax the notion of principal normality. Indeed, for operators with simple characteristics we introduce a notion of singular principal normality and we prove that it ensures compact uniqueness. In a second part we show that this condition is relevant in proving that if it is violated in a strong sense then non uniqueness holds for a zero$^{\text{th}}$ order perturbation of the operator.

The uniqueness result uses Carleman estimates with singular weights, which are proved by the method introduced by Lerner [L] in the standard case. The main difficulty is then that the proof requires a Fefferman-Phong inequality for pseudo-differential operators with symbols in a non temperate class in the sense of Hörmander [H]. However this inequality has been established in a recent paper by the authors [CDZ]. The proof of the non uniqueness result uses the method developed in earlier works (see [A], [Z]) with new difficulties related to the singularities.

2. Statement of the results

Let P be a homogeneous differential operator of order $m \geq 1$, with complex valued C^∞ coefficients, in a neighborhood V of a point x_0 in $\mathbf{R}^n$, and symbol p. Let S be a C^∞ hypersurface through x_0, given in V by $S \cap V = \{x \in V: \varphi(x) = 0\}$, with $\varphi \in C^\infty$ and $\varphi'(x) \neq 0$ in $S \cap V$. We shall set $V^+ = \{x \in V: \varphi(x) > 0\}$.

The symbol p will be assumed to have simple complex roots

$$\text{(H.1)} \quad \begin{cases} \xi \in \mathbf{R}^n,\ \tau \in \mathbf{R} \quad \text{and} \quad p(x_0, \xi + i\tau\varphi'(x_0)) = \{p, \varphi\}(x_0, \xi + i\tau\varphi'(x_0)) = 0 \\ \text{imply } \xi = \tau = 0\ . \end{cases}$$

Here $\{,\}$ is as usual the Poisson bracket.

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