J. Math. Kyoto Univ. (JMKYAZ)
36-2 (1996) 311-329

Minimization of the embeddings of the curves into the affine plane

By

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0. Introduction

Let C be a smooth affine algebraic curve with only one place at infinity defined over an algebraically closed field k of characteristic zero; we also call C a *once punctured* smooth algebraic curve. Assume that C is embedded into the affine plane $\mathbf{A}^2$ as a closed curve. The image of C by an algebraic automorphism of $\mathbf{A}^2$ is again a curve of the same nature as C. One may then ask what is the smallest among the degrees of $\varphi(C)$ when φ ranges over automorphisms of $\mathbf{A}^2$. We say that $\varphi(C)$ is a *minimal embedding* of C if the degree of $\varphi(C)$ is the smallest.

The question was first treated by Abhyankar-Moh [1] and Suzuki [12] in the case of genus g of C is zero. Namely, *a minimal embedding of the affine line is a coordinate line.* The cases $g = 2, 3, 4, \ldots$ were treated by Neumann [8] by topological methods and by A'Campo-Oka [3] depending on Tschirnhausen resolution tower.

We shall here propose a different algebro-geometric approach based on the classification of degenerations of curves, which enables us to describe an automorphism φ of $\mathbf{A}^2$ minimizing the degree of $\varphi(C)$.

Our theorem is the following:

Theorem. *Let C be a once punctured smooth algebraic curve of genus g, which is embedded into the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$ as a closed curve defined by $f(x, y) = 0$. Then there exists new coordinates u, v of $\mathbf{A}^2$ such that*

(1) $k[x, y] = k[u, v]$, *and*

(2) $h(u, v) := f(x(u, v), y(u, v))$ *and* $e = \deg h(u, v)$ *are given as follows if* $g \leq 4$;

Case $g = 0$: $e = 1$ *and* $h = u$.

Case $g = 1$: $e = 3$ *and* $h = v^2 - (u^3 + au + b)$ *with* $a, b \in k$.

Case $g = 2$: $e = 5$ *and* $h = v^2 - (u^5 + au^3 + bu^2 + cu + d)$ *with* $a, b, c, d \in k$.

Case $g = 3$: *There are three types*:

(1) $e = 4$ *and* $h = v^3 + g_1(u)v - (u^4 + g_2(u))$ *with* $g_i(u) \in k[u]$ *and* $\deg g_i(u) \leq 2$ *for* $i = 1, 2$.

Communicated by Prof. K. Ueno, March 3, 1995