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Homotopy-commutativity in rotation groups

By

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1. Introduction

Assume G is a topological group and S, S' are subspaces of G, each of which contains the unit as its base point. There is the commutator map c from $S \wedge S'$ to G which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$. We say S and S' homotopy-commute in G if c is null homotopic.

In this paper, we describe the homotopy-commutativity of the case $G = SO(n+m-1)$, $S = SO(n)$ and $S' = SO(m)$ where $n, m > 1$. Here we use the usual embeddings

$$SO(1) \subset SO(2) \subset SO(3) \subset \cdots .$$

Trivially $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n+m)$. And it is known that if $n+m>4$, $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n+m-2)$. (See [1] and [2].) But the homotopy-commutativity in $SO(n+m-1)$ has not been solved exactly.

We shall say a pair (n, m) is irregular if $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n+m-1)$, and regular if they do not. In [1] the following problem is proposed; "when is (n, m) irregular?", and the next theorem is showed.

Theorem 1.1 (James and Thomas). *Let $n+m \neq 4, 8$. If n or m is even or if $d(n)=d(m)$ then (n, m) is regular, where $d(q)$, for $q \geq 2$, denotes the greatest power of 2 which devides $q-1$.*

In this paper we shall prove the more strict result as showed in the next theorem.

Theorem 1.2. *If n or m is even or if $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$ then (n, m) is regular.*

We identify $\mathbf{RP}^{k-1} \overset{i_k}{\hookrightarrow} SO(k)$ by the following way. Let $i'_k\colon \mathbf{RP}^{k-1} \to O(k)$ be the map which attaches a line $l \in \mathbf{RP}^{k-1}$ with $i'_k(l) \in O(k)$ defined by

$$i'_k(l)(v) = v - 2(v, e)e,$$

where e is a unit vector of l and $v \in \mathbf{R}^k$. And let $i_k(l) = i'_k(l_0)^{-1} \cdot i'_k(l)$ where l_0 is the base point of $\mathbf{RP}^{k-1}$. Then i_k preserves the base points.

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