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# Donsker's delta functions and approximation of heat kernels by the time discretization methods

By

Yaozhong HU* and Shinzo WATANABE

## Introduction

Time discretization approximation schemes for solutions of stochastic differential equations have been studied by many people and are treated, e.g., in the book of Kloeden-Platen [Kl-Pl92]. Since heat kernels are the probability densities of the law of solutions, it might be worth-while to ask if these approximation schemes provide a natural scheme of approximation for heat kernels. Purpose of this paper is to propose one of such schemes with a help of Malliavin calculus.

In section 1, we introduce the notion of Donsker's delta functions as a class of generalized Wiener functionals on Wiener space. In section 2, we obtain a general approximation result for Donsker's delta functions. In section 3, we consider the case of Wiener functionals given by solutions to stochastic differential equations. An Itô-Taylor approximation scheme of order $\gamma$ for the solution has been introduced by Kloeden and Platen [Kl-Pl95]. Here we improve their result of the strong convergence in the $L_2$-norm to the strong convergence in every Sobolev norm in the Malliavin calculus (Theorem 3.1). This is a main result of this paper and its proof is given in section 4. This result, combined with general results in section 2, yields some strong approximation scheme for Donsker's delta functions and thereby an approximation result for the heat kernel in the form of Theorem 3.2. However, it should be remarked that the heat kernel is given by a generalized expectation of Donsker's delta function and therefore, what is involved in this problem is essentially an weak approximation. The rate of convergence in Theorem 3.2 is that of the strong approximation and it can be improved to the rate of weak convergence. For such improvements, we refer to the recent works by Bally and Talay [B-T95] and Kohatsu-Higa [Ko95].

## 1. Malliavin calculus and Donsker's delta functions

Let $(W, H, P)$ be a (classical or abstract) Wiener space, where $H$ is the Cameron-Martin Hilbert space and $P$ is the Wiener measure. Let $F: W \to \mathbf{R}^d$

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