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Extending local representations to global representations

By

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1. Introduction

It is a theorem of Deligne (and Deligne-Serre for weight 1) that for a cuspidal eigenform of the Hecke operators on the upper half plane which is of weight k, the eigenvalues of the Hecke operators T_p are algebraic integers a_p with $|a_p| \leq 2p^{(k-1)/2}$. In §2 of this note we pose a converse question to this, and analyse to what extent CM forms can be used to answer it. In §3 an analogous issue is considered in the setting of Galois representations which can be thought of as the non-abelian analogue of the Grunwald-Wang theorem in Class Field Theory. We may view these questions (cf. the question of §2 and Remark 4 of §3) as asking for a kind of Chinese Remainder Theorem in the setting of automorphic and Galois representations respectively. In §4 we use the cohomology of modular curves to construct automorphic representations of $PGL_2(\mathbf{Q})$ with given local component at p and unramified outside p.

2. Chinese remainder theorem for automorphic representations

The aim of this section is to pose the following question and provide an answer to it in some very particular cases.

QUESTION. Suppose that we are given finitely many primes $p_1, \ldots, p_r$, and algebraic integers α_i for every i, $1 \leq i \leq r$, which have the property that $\sigma(\alpha_i)\overline{\sigma(\alpha_i)} = p_i^{k-1}$ for some integer $k \geq 1$ and for every embedding $\sigma\colon \overline{\mathbf{Q}} \to \mathbf{C}$. Then does there exist a cusp form f of weight k which is an eigenform of all the Hecke operators such that the Euler factor at p_i of the L-series of f, for every i, $1 \leq i \leq r$, is

$$L_{p_i}(f, s) = \frac{1}{\left(1 - \frac{\alpha_i}{p_i^s}\right)\left(1 - \frac{\bar{\alpha}_i}{p_i^s}\right)}?$$

The recent work of Wiles and Taylor on the Shimura-Taniyama conjecture, cf. [W] and [TW], and its subsequent refinement by Diamond, cf. [D], proves

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