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Cancellation of lattices and approximation properties of division algebras

By

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§0. Introduction

Let R be a Dedekind domain with the quotient field K. Let Λ be an R-order. In this general setting, it is proved in [3] that Roiter-Jacobinski type Divisibility Theorem holds for Λ-lattices. As a consequence, for a Λ-lattice L, the following two cancellation properties are equivalent.

(c) If L' is a local direct summand of $nL=L\oplus\cdots\oplus L$ for some $n\geq 0$, then $L\oplus L'\simeq M\oplus L'$ implies $L\simeq M$.

(c′) If $L\oplus nL\simeq M\oplus nL$ for some $n\geq 0$, then $L\simeq M$.

As was pointed out in [3], putting $\Gamma:=\mathrm{End}_\Lambda L$ and $B:=K\Gamma$, there is an intimate connection between cancellation property and the approximation property of the group of Vaserstein $\widetilde{E}(\widehat{B})$ in the idele topology of $\widehat{B}^\times$, of which precise definitions will be recalled in §1.

Here we only indicate, $\widehat{R}:=\Pi R_p$, the direct product of p-adic completions over all maximal ideals of R, $\widehat{M}:=M\otimes_R\widehat{R}$ for any R-alegbra M, and $\widetilde{E}(C):=\langle(1+xy)(1+yx)^{-1}|x,y\in C, 1+xy\in C^\times\rangle$ for any ring $C\ni 1$. Our first remark is

Proposition 1 *(proof in 1.5). The property (c′) for L is equivalent with the following property (c″) of* Γ.

(c″) $\widetilde{E}(\widehat{B})\subset\widehat{\Gamma}^\times B^\times$ *as subsets of* $\widehat{B}^\times$.

0.1. We shall consider, for any finite dimensional K-algebra B, the following three *approximation properties over* R, in the idele topology of $\widehat{B}^\times$.

(a) Strong approximation property :

$\widetilde{E}(B)$ is dense in $\widetilde{E}(\widehat{B})$

(a′) $B^\times$-approximation property :

$\widetilde{E}(\widehat{B})$ is contained in the closure of $B^\times$.

(a″) $\widehat{R}^\times B^\times$-approximation property :

$\widetilde{E}(\widehat{B})$ is contained in the closure of $\widehat{R}^\times B^\times$.

There are the obvious implications (a) ⇒ (a′) ⇒ (a″). Our second (rather

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