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On the zeros of the Epstein zeta functions

By

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§ 1. Introduction

The purpose of the present article is to extend our recent results [8] concerning the distribution of the zeros of the Epstein zeta functions. We have shown there that the "k-analogue" of "the GUE law" fails for the Epstein zeta functions $\zeta(s, Q)$. A more precise definition of "the GUE law" and also the meaning of the "k-analogue" will be given below. The Epstein zeta function $\zeta(s, Q)$ is defined by

$$\zeta(s, Q) = \frac{1}{2}\sum_{x,y}{}' Q(x, y)^{-s} \text{ for } \Re(s) > 1,$$

where x, y runs over all integers excluding $(x, y) = (0, 0)$, $s = \sigma + it$ with real numbers σ and t, $Q(x, y) = ax^2 + bxy + cy^2$ is a positive definite quadratic form with discriminant $\Delta = b^2 - 4ac$, a, b and c are real numbers and $a > 0$ and we put

$$k = \frac{\sqrt{|\Delta|}}{2a}.$$

Some of the well known results concerning $\zeta(s, Q)$ will be recalled below. In the present article, we are concerned with the distribution of the zeros of the Epstein zeta functions associated with the positive definite quadratic forms of more variables. However, we shall treat only the simpler cases among them, for simplicity. We shall also give some new results concerning the simplest $\zeta(s, Q)$. A further extension is possible and will appear elsewhere.

Let d be a positive number. Here we are mainly concerned with the Epstein zeta functions of the form

$$G_d(s) = \sum{}' \frac{1}{(m_1^2 + m_2^2 + d(m_3^2 + m_4^2))^s},$$

where $\Re(s) > 2$, the dash indicates that m_j's run over the integers excluding the case $(m_1, m_2, m_3, m_4) = (0, 0, 0, 0)$. We are particularly intersted in the distribution of the zeros of $G_d(s)$. We put for a convenience

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