A Bifurcation phenomenon for the periodic solutions of the Duffing equation

By

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1. Introduction and Result

In this paper, we study a bifurcation phenomenon for the periodic solutions of the following Duffing equation which describes a nonlinear forced oscillation:

\[ u''(t) + \mu u'(t) + \kappa u(t) + \alpha u^3(t) = f(t), \quad t \in \mathbb{R}, \]

where \( \mu \) and \( \alpha \) are positive constants, \( \kappa \) is a nonnegative constant, and \( f(t) \) is a given periodic external force. It is known that for any periodic external force there exists at least one periodic solution of (1.1) with the same period as the external force. Furthermore, if the external force is suitably small, then the periodic solution is proved to be unique and asymptotically stable. On the other hand, in the case of the relatively large external force, numerical computations show a possibility of not only the non-uniqueness of the periodic solution but also the existence of various bifurcation phenomena. In particular, a strange attractor discovered by Ueda [6], so called Japanese attractor, is well known. However, it is surprising that there have been no mathematical proofs of the existence of bifurcation for the periodic solutions of (1.1). The aim of this paper is to give a mathematical proof of the existence of bifurcation for a special family of external forces. To do that, we define the one-parameter families of periodic functions \( \{u_i(t)\}_{i>0} \) and \( \{f_i(t)\}_{i>0} \) with period one by

\[
\begin{align*}
  u_i(t) : &= \lambda \sin 2\pi t, \quad \lambda > 0, \\
  f_i(t) : &= u''_i(t) + \mu u'_i(t) + \kappa u_i(t) + \alpha u^3_i(t),
\end{align*}
\]

so that the equation (1.1) has the trivial periodic solution \( u(t) = u_i(t) \) to the external force \( f(t) = f_i(t) \) for any \( \lambda > 0 \). Then our main Theorem is

**Theorem 1.** Suppose \( \mu \) and \( \kappa \) satisfy