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The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action

By

Hiroaki HAMANAKA* and Shin-ichiro HARA

1. Introduction

Let p be a prime number and G be a compact, connected, simply connected and simple Lie group. Let ΩG be the loop space of G. Bott showed $H_*(\Omega G;Z/p)$ is a finitely generated bicommutative Hopf algebra concentrated in even degrees, and determined it for classical groups G ([1]).

Here, let G be an exceptional Lie group, that is, $G=G_2, F_4, E_6, E_7, E_8$. In [2], K. Kozima and A. Kono determined $H_*(\Omega G; Z/2)$ as a Hopf algebra over $\mathcal{A}_2$, where $\mathcal{A}_p$ is the mod p Steenrod Algebra and acts on it dually.

Let Ad : $G\times G\to G$ and ad : $G\times\Omega G\to\Omega G$ be the adjoint actions of G on G and ΩG respectively. In [3], the cohomology maps of these adjoint actions are studied and it is shown that $H^*(ad\ ;Z/p)=H^*(p_2;Z/p)$ where p_2 is the second projection if and only if $H^*(G;Z)$ is p-torsion free. For $p=2$, 3 and 5, some exceptional Lie groups have p-torsions on its homology. Moreover in [8, 9] mod p homology map of ad is determined for $(G, p)=(G_2, 2)$, $(F_4, 2)$, $(E_6, 2)$, $(E_7, 2)$ and $(E_8, 5)$. This result is applied to compute the $\mathcal{A}_5$ module structure of $H_*(\Omega E_8; Z/5)$ and $H^*(E_8; Z/5)$ in [9].

For a compact and connected Lie group G, the free loop group of G is denoted by $LG(G)$, i. e. the space of free loops on G equipped with multiplication as

$$\phi\cdot\psi(t)=\phi(t)\cdot\psi(t),$$

and has ΩG as its normal subgroup. Then

$$LG(G)/\Omega G\cong G,$$

and identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semi-direct product of G and ΩG. This means that the homology of $LG(G)$ is determined by the homology of G and ΩG as module and the algebra structure of $H_*(LG(G); Z/p)$ depends on $H_*(ad\ ; Z/p)$ where

$$ad : G\times\Omega G\to\Omega G$$

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