The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action

By

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1. Introduction

Let \( p \) be a prime number and \( G \) be a compact, connected, simply connected and simple Lie group. Let \( \Omega G \) be the loop space of \( G \). Bott showed \( H_\ast (\Omega G; \mathbb{Z}/p) \) is a finitely generated bicommutative Hopf algebra concentrated in even degrees, and determined it for classical groups \( G \) ([1]).

Here, let \( G \) be an exceptional Lie group, that is, \( G = G_2, F_4, E_6, E_7, E_8 \). In [2], K. Kozima and A. Kono determined \( H_\ast (\Omega G; \mathbb{Z}/2) \) as a Hopf algebra over \( \mathcal{A}_p \), where \( \mathcal{A}_p \) is the mod \( p \) Steenrod Algebra and acts on it dually.

Let \( \text{Ad} : G \times G \to G \) and \( \text{ad} : G \times \Omega G \to \Omega G \) be the adjoint actions of \( G \) on \( G \) and \( \Omega G \) respectively. In [3], the cohomology maps of these adjoint actions are studied and it is shown that \( H^\ast (\text{ad} ; \mathbb{Z}/p) = H^\ast (p_2 ; \mathbb{Z}/p) \) where \( p_2 \) is the second projection if and only if \( H^\ast (G ; \mathbb{Z}) \) is \( p \)-torsion free. For \( p = 2, 3 \) and \( 5 \), some exceptional Lie groups have \( p \)-torsions on its homology. Moreover in [8, 9] mod \( p \) homology map of \( \text{ad} \) is determined for \( (G, p) = (G_2, 2), (F_4, 2), (E_6, 2), (E_7, 2) \) and \( (E_8, 5) \). This result is applied to compute the \( \mathcal{A}_5 \) module structure of \( H_\ast (\Omega E_6; \mathbb{Z}/5) \) and \( H_\ast (E_7; \mathbb{Z}/5) \) in [9].

For a compact and connected Lie group \( G \), the free loop group of \( G \) is denoted by \( LG (G) \), i.e. the space of free loops on \( G \) equipped with multiplication as

\[
\phi \cdot \psi (t) = \phi (t) \cdot \psi (t),
\]

and has \( \Omega G \) as its normal subgroup. Then

\[
LG (G) / \Omega G \cong G.
\]

and identifying elements of \( G \) with constant maps from \( S^1 \) to \( G \), \( LG (G) \) is equal to the semi-direct product of \( G \) and \( \Omega G \). This means that the homology of \( LG (G) \) is determined by the homology of \( G \) and \( \Omega G \) as module and the algebra structure of \( H_\ast (LG (G); \mathbb{Z}/p) \) depends on \( H_\ast (\text{ad} ; \mathbb{Z}/p) \) where

\[
\text{ad} : G \times \Omega G \to \Omega G
\]

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