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Cyclic automorphic forms on a unitary group

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1. Statement of results

Let $\mathbf{C}$ be an algebraic subgroup of an algebraic group $\mathbf{G}$ over a global field F, and denote by ϕ: $\mathbf{G}(F)\backslash\mathbf{G}(\mathscr{A})\to\mathscr{C}$ a cusp form in the cuspidal representation π of the adele group $\mathbf{G}(\mathscr{A})$, where $\mathscr{A}$ is the ring of F-adeles. Denote by $\mathbf{Z}$ the center of $\mathbf{G}$ and by $\omega=\omega_\pi$: $\mathbf{Z}(\mathscr{A})/\mathbf{Z}(F)\to\mathscr{C}^\times$ the central character of π. Suppose that the homogeneous space $\mathbf{C}(F)\backslash\mathbf{C}(\mathscr{A})$ has finite volume; we call it a *cycle*. Fix a unitary character ξ of $\mathbf{C}(F)\backslash\mathbf{C}(\mathscr{A})$ in $\mathscr{C}^\times$. Signify by $P(\phi)=P_{\mathbf{C},\xi}(\phi)=\int_{\mathbf{C}(F)\backslash\mathbf{C}(\mathscr{A})}\phi(c)\overline{\xi}(c)dc$ the *ξ-period* of the form ϕ on the cycle $\mathbf{C}(F)\backslash\mathbf{C}(\mathscr{A})$ of $\mathbf{G}(F)\backslash\mathbf{G}(\mathscr{A})$. Here $\overline{\xi}(c)$ is the complex conjugate of $\xi(c)$. We simply say "period" when $\xi=1$.

Motivated for example by classical questions on the L^2-cohomology of bounded symmetric spaces of the form $\Gamma\backslash G/C$, we wish to determine the *ξ-cyclie* cuspidal $\mathbf{G}(\mathscr{A})$-modules π; these are the cuspidal π for which there is a form $\phi\in\pi\subset L^2_\omega(\mathbf{G}(F)\backslash\mathbf{G}(\mathscr{A}))$ with a non-zero ξ-period $P(\phi)$ (or $P_{\mathbf{C},\xi}(\phi)$ if the dependence on $\mathbf{C}$ and ξ needs to be made explicit). The interesting phenomenon which occurs in this context is, that in order to by cyclic, a global representation needs—in addition to being locally cyclic at all places—to overcome a purely global obstruction.

Such a question was studied first by Waldspurger [W1,2] when $\mathbf{G}=PGL(2)=SO(3)$ and $\mathbf{C}=SO(2)$ (=elliptic torus of $\mathbf{G}$ which splits over a quadratic extension of the base field) on using the Weil representation, then by Harder-Langlands-Rapoport [HLR] when $\mathbf{G}$ is $GL(2)$ over a quadratic extension E of F and $\mathbf{C}$ is $GL(2)$ over F, and then by Jacquet-Lai [JL] and Jacquet [J1,2] who introduced a "relative trace formula". The case of $\mathbf{G}=SO(4)\times SO(3)$ and $\mathbf{C}=SO(3)$ (and ξ is a form in a cuspidal, not one-dimensional, automorphic representation of $\mathbf{C}(\mathscr{A})$), was studied by (D. Prasad [P1,2] locally and) Harris-Kudla [HK], again on using the Weil representation. The analogue of the trace formula technique was later used in a series of cases, where $\mathbf{G}=GL(2,E)$ [F5], or where $\mathbf{G}=GL(n,E)$ [F6], and $\mathbf{C}=GL(n,F)$, with E/F being a quadratic extension of global (or local) fields. This last case is related to base-change for the unitary group. Local aspects of the dual case, where $\mathbf{G}=GL(n,E)$ and $\mathbf{C}$ is a unitary group—which is related

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