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# Projective elements in $K$-theory and self maps of $\Sigma CP^\infty$

By

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## 1. Introduction and statements of results

In this paper, we will work in the homotopy category of based spaces and based maps. Given a space $X$, we denote the reduced $K$-theory by $K(X)$ and the homology group of integral coefficients by $H_*(X)$. Let $CP^\infty$ be the infinite dimensional complex projective space. Let $\eta$ be the canonical line bundle over $CP^\infty$ and $i$: $CP^\infty \to BU$ be the classifying map of the virtual bundle $\eta - 1$. Since $BU$ has a loop space structure which is derived from the Whitney sum of complex vector bundles, there exists a unique extension of $i$ to the loop map $j$: $\Omega\Sigma CP^\infty \to BU$.

In this paper we investigate the following problems:

Given an element $\alpha \in K(X)$, when does there exist a lift $\widehat{\alpha} \in [X, \Omega\Sigma CP^\infty]$ such that $j_*(\widehat{\alpha}) = \alpha$? If $\alpha$ has a lift, how we can construct the lift $\widehat{\alpha}$?

Define

$$PK(X) = \{\alpha \in K(X) \mid \exists\ \widehat{\alpha} \in [X, \Omega\Sigma CP^\infty] \text{ such that } j_*(\widehat{\alpha}) = \alpha\}.$$

If an element $\alpha \in K(X)$ belongs to $PK(X)$, we call that $\alpha$ is projective.

The significance of the above problem is as follows:

The James splitting theorem [2] implies that there exists a loop map $\theta$: $BU \to \Omega^\infty\Sigma^\infty CP^\infty$ such that the following diagram commutes:

$$\begin{array}{ccc} \Omega\Sigma CP^\infty & \xrightarrow{E^\infty} & \Omega^\infty\Sigma^\infty CP^\infty \\ \downarrow j & \nearrow \theta & \\ BU. & & \end{array}$$

Therefore, given an element $\alpha \in K(X)$, we have the stable map, $adj.(\theta(\alpha))$: $\Sigma^\infty X \to \Sigma^\infty CP^\infty$. Using the information of $K(X)$, we can calculate the induced homomorphism [3], [4] of $adj.(\theta(\alpha))_*$: $H_*(X) \to H_*(CP^\infty)$. If $\alpha$ has a lift $\widehat{\alpha}$, then this implies that the stable map $adj.(\theta(\alpha))$ and its induced

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