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A generalization of the parallelogram equality in normed spaces

By

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Let $(X, \|\cdot\|)$ be a real normed space. Then on X^2 there always exist the functionals:

$$\tau_{\pm}(x, y) := \lim_{t \to \pm 0} t^{-1}(\|x+ty\| - \|x\|) \qquad (x, y \in X). \tag{1}$$

$$g(x, y) := \frac{\|x\|}{2}(\tau_{-}(x, y) + \tau_{+}(x, y)) \qquad (x, y \in X)^{1)}. \tag{2}$$

The functional g is a natural generalization of the inner product $(\cdot, \cdot)$, which follows from its properties:

$$g(x, x) = \|x\|^2 \qquad (x \in X), \tag{3}$$

$$g(\alpha x, \beta y) = \alpha\beta g(x, y) \qquad (x, y \in X; \alpha, \beta \in R), \tag{4}$$

$$g(x, x+y) = \|x\|^2 + g(x, y) \qquad (x, y \in X), \tag{5}$$

$$|g(x, y)| \leq \|x\|\|y\| \qquad (x, y \in X), \tag{6}$$

$(X, \|\cdot\|)$ is an inner product space if and only if $g(x, y)$ is an inner product of vectors x and y, for all $x, y \in X$. (7)

By use of the functional g, we may define many geometrical points in normed spaces (angle between two vectors, the projection of the vector x on the vector y, many types of orthogonalities, orthonormal system, and so on) (cf. [2] to [5]).

In an inner product space X the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 + \|y\|^2)(x, y) \qquad (x, y \in X) \tag{8}$$

holds, which is equivalent to the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad (x, y \in X). \tag{9}$$

In normed spaces, the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 g(x, y) + \|y\|^2 g(y, x)), \qquad (x, y \in X) \tag{10}$$

is a generalization of the equality (8).

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1) The notation g is according to the name Gâteaux.