BGG-resolution for $\alpha$-stratified modules over simply-laced finite-dimensional Lie algebras

By

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1. Introduction

This paper is a sequel of [6] where the submodule structure of $\alpha$-stratified (i.e. torsion free with respect to the subalgebra corresponding to a root $\alpha$) generalized Verma modules was studied. The results obtained there generalize the classical theorem of Bernstein-Gelfand-Gelfand on Verma module inclusions. The crucial role in the study is played by the generalized Weyl group $W_\alpha$ that acts on the space of parameters of generalized Verma modules.

Let $G$ be a simple finite-dimensional Lie algebra over the complex numbers with a simply-laced Coxeter-Dynkin diagram (i.e. there are no multiple arrows). In the present paper for any such algebra we construct a strong BGG-resolution for $\alpha$-stratified irreducible modules in the spirit of [1,10]. The non-simply-laced case is more complicated (cf. [6]). In particular, the proof of the crucial Theorem 4 is based on the fact that the diagram is simply-laced.

The structure of the paper is the following. In Section 2 we collect the notation and preliminary results. A weak generalized BGG-resolution is constructed in Section 3. Here we follow closely [1]. Section 4 contains an extension lemma for $\alpha$-stratified modules which generalizes a well-known result of Rocha-Caridi for Verma modules [10]. Our proof is analogous to the one of Humphreys for Verma modules [8]. In Section 5 we study the maximal submodule of the generalized Verma module and construct a strong generalized BGG-resolution for $\alpha$-stratified irreducible modules in Section 6. Finally, in Section 7 we give a character formula for certain $\alpha$-stratified irreducible modules.

2. Notation and preliminary results

Let $C$ denote the complex numbers, $\mathbb{Z}$ all integers, $\mathbb{N}$ all positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let $H$ be a Cartan subalgebra of $G$ and $\Delta$ the root system of $G$.

Let $\pi$ be a basis of $\Delta$ containing $\alpha$, $\Delta_+ = \Delta_+(\pi)$ the set of positive (negative) roots with respect to $\pi$. For any $S \subset \pi$ let $\Delta_+(S)$ be a subset generated by $S$ (it