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On explicit constructions of rational elliptic surfaces with multiple fibers

By

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This note is a supplement to our previous paper [F], where I studied the basic property of rational elliptic surfaces with multiple fibers S through the logarithmic transformations. It is also a nine-points-blowing-up of $\mathbf{P}^2$, but any (-1)-curve is a multi-section of its elliptic fibering over $\mathbf{P}^1$. If the nine points P_i $(1 \leq i \leq 9)$ on $\mathbf{P}^2$, which are the center of blowing-ups, are mutually distinct and the multiple fiber is of type ${}_mI_0$, it is obtained from the pencil generated by m-fold cubic which passes through p_i's and an irreducible curve of degree $3m$ which has an ordinary singularity of multiplicity m at each p_i and is non-singular outside them. And the anti-pluricanonical map $\Phi_{|-mK_S|} : S \to \mathbf{P}^1$ gives the unique structure of an elliptic fibration. Such a pencil (called Halphen pencil) already appeared in [Nag], §4, Theorem (1), case (ㅅ), when Nagata constructed a rational surface with infinitely many (-1)-curves. Also Hironaka and Matsumura [H-M] applied it to construct examples of a curve C in a smooth projective surface F, where C satisfies $G1$ conditions in F, but not $G2$ conditions. On the other hand, when part of the nine points p_i's on $\mathbf{P}^2$ are infinitely near, the Halphen pencil degenerates into a more complicated one. Any (-1)-curve e on S is an m-sheeted covering of the base curve $\mathbf{P}^1$, branching over the point where the multiple fiber lie with the ramification index m. Hence, it is not at all easy to find nine (-1)-curves on S, see how they intersect the irreducible components of each singular fiber and repeat blowing-downs to $\mathbf{P}^2$.

Here, we shall describe an *explicit* construction of rational elliptic surfaces with multiple fibers through the 'Halphen transform' in the sense of [H-L], which is some kind of *birational transformations.*

We recall the following result.

Theorem (A) ([F], [H-L]). *Let C be a non-singular cubic (resp. a nodal cubic) in $\mathbf{P}^2$ with the fixed inflexion point Q on C such that C should be given the natural group structure with Q as the identity. Take nine points p_i $(1 \leq i \leq 9)$ on C (which may be infinitely near) and let S be the surface obtained by blowing up $\mathbf{P}^2$ at p_i's $(1 \leq i \leq 9)$. Then S has the structure of an elliptic surface with one multiple fiber of multiplicity m if and only if $\sum_{i=1}^{9} p_i$ is of order m in the elliptic curve (resp.*

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