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Some categories of lattices associated to a central idempotent

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0. Let R be a noetherian integral domain with field of quotients K. An *R-lattice* is a finitely generated torsion free R-module. An *R-order* is an R-algebra Λ which is an R-lattice. For an R-order Λ, a *Λ-lattice* is a left Λ-module which is an R-lattice. Let lat Λ denote the category of Λ-lattices.

Let e be a central idempotent of the K-algebra $\tilde{\Lambda} := K \otimes_R \Lambda$, so that $e\Lambda$ is an R-order in the K-algebra $e\tilde{\Lambda}$. The category lat $e\Lambda$ can be viewed as a full subcategory of lat Λ via the ring homomorphism $\Lambda \to e\Lambda$, $(\lambda \mapsto e\lambda)$.

0.0. A purpose of this paper is to investigate the *quotient category* $\mathscr{C} := \operatorname{lat}\Lambda/\operatorname{lat}e\Lambda$. By definition, $\mathscr{C}$ has the same objects as $\operatorname{lat}\Lambda$, and $\operatorname{Hom}_{\mathscr{C}}(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/I(X, Y)$, where $I(X, Y)$ is the totality of Λ-morphisms $f: X \to Y$ which factor through some object of lat $e\Lambda$. By 2.1.1, $\operatorname{Hom}_{\mathscr{C}}(X, Y) = (1-e)$ $\operatorname{Hom}_\Lambda(X, Y)$ holds.

Let $\mathscr{P}$ be the full subcategory of $\mathscr{C}$ formed by $X \in \mathscr{C}$ satisfying the following condition (∗).

(∗) There exist a projective Λ-lattice P, $e\Lambda$-lattice Ω and an exact sequence $0 \to \Omega \to P \to X \to 0$ in $\operatorname{lat}\Lambda$.

0.1. Theorem (Proof in 2.5). *Assume that $\mathscr{P}$ has an additive generator Q (i.e. any object in $\mathscr{P}$ is isomorphic to a direct summand of $Q^n = Q \oplus \cdots \oplus Q$ for some n). Put $\Gamma := \operatorname{Hom}_{\mathscr{C}}(Q, Q)$, $FX := \operatorname{Hom}_{\mathscr{C}}(Q, X)$ for $X \in \operatorname{lat}\Lambda$. Then Γ is an R-order and F induces a categorical equivalence from $\mathscr{C} = \operatorname{lat}\Lambda/\operatorname{lat}e\Lambda$ to $\operatorname{lat}\Gamma$.*

0.2. Assume that R is a complete discrete valuation ring. Then $\operatorname{lat}\Lambda$ is a Krull-Schmidt category, and any $X \in \operatorname{lat}\Lambda$ has a projective cover $0 \to \Omega(X) \to P(X) \to X \to 0$. In this case, the above $\mathscr{P}$ can be described as $\{X \in \mathscr{C} \mid \Omega(X) \in \operatorname{lat}e\Lambda\}$.

Let $\operatorname{ind}\Lambda$ denote the set of isomorphism classes of indecomposable Λ-lattices and put

$$\mathscr{Q} := \{X \in \operatorname{ind}\Lambda - \operatorname{ind}e\Lambda \mid \Omega(X) \in \operatorname{lat}e\Lambda\}.$$

If $\mathscr{Q}$ is a finite set, by the additivity of projective cover, $\mathscr{P}$ has an additive generator $Q = \bigoplus_{X \in \mathscr{Q}} X$.

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