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1-cocycles on the group of diffeomorphisms

By

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§1. Introduction

Let M be a d-dimensional paracompact C^{∞}-manifold and $\mathrm{Diff}(M)$ be the group of all C^{∞}-diffeomorphisms on M. Among the subgroups of $\mathrm{Diff}(M)$, we take here the group $\mathrm{Diff}_0(M)$ which consists of all $g \in \mathrm{Diff}(M)$ with compact supports, that is the set $\{P \in M | \ g(P) \neq P\}$ is relatively compact. Up to the present time, unitary representations $(U, \mathscr{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups ($\mathscr{H}$ is the representation Hilbert space of U) are constructed and considered by many authors, for example [4], [5], [6], [7], [8], [9], [10], [12] and [19]. The first purpose of this paper is a trial to construct some differentiable method to analyze these representations $(U, \mathscr{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups. Roughly speaking, we wish to consider a differential representation of a given one. So the first step we should do is to define a suitable Lie algebra $\mathscr{G}_0$ of $\mathrm{Diff}_0(M)$, regarding it as an infinite dimensional Lie group. For the case of compact manifold, it is well known for a pretty long time ago that $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ is an infinite dimensional Lie group whose modelled space is a Fréchet space called strong inductive limit of Hilbert spaces by a few authors. (cf [13]). So after them, we are naturally derived that we should take the set $\Gamma_0(M)$ of all C^{∞}-vector fields X with compact supports as the Lie algebra $\mathscr{G}_0$, and it is appropriate to take the map $\mathrm{Exp}(X)$ as the exponential map exp from $\Gamma_0(M)$ to $\mathrm{Diff}_0(M)$, where $\{\mathrm{Exp}(tX)\}_{t \in \boldsymbol{R}}$ is the 1-parameter transformation group generated by $X \in \Gamma_0(M)$. Thus formally we have self adjoint operators $dU(X)$ on $\mathscr{H}$ by Stone's result,

$$U(\mathrm{Exp}(tX)) = \exp(\sqrt{-1}\, t\, dU(X)),$$

and simaltaneouly there arise many problems for such $dU(X)$ and for Exp maps. Among them the following questions are fundamental.

(1) Does $\sqrt{-1}dU$ become a linear representation under suitable restrictions of the domain of each $dU(X)$?

(2) Is the common domain of $\{dU(X)\}_{X \in \Gamma_0(M)}$ rich such one like Gårding space?

(3) Is the subgroup generated by $\mathrm{Exp}(X)$, $X \in \Gamma_0(M)$ dense in $\mathrm{Diff}_0(M)$?

It is easily expected that the linearity of $\sqrt{-1}dU$ mostly depends on the usual

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