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Limit theorem for symmetric statistics with respect to Weyl transformation: Disappearance of dependency

By

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1. Introduction

It is known that there are several kinds of deterministic sequences $\{x_n\}_{n=1}^{\infty}$ on $\boldsymbol{T}^m=[0,1)^m$ having the following property : For any function $F:\boldsymbol{T}^m\to\boldsymbol{R}$ of finite variation, we have

$$\left|\int_{\boldsymbol{T}^m} F(x)dx-\frac{1}{N}\sum_{n=1}^{N} F(x_n)\right|=O(N^{-1+\varepsilon}),\qquad N\to\infty.\quad (^\forall\varepsilon>0) \tag{1}$$

These sequences are called *low discrepancy sequences* ([2]). The convergence (1) can be used for numerical integrations in $\boldsymbol{T}^m$, which is called the *quasi Monte Carlo method.* Since the usual Monte Carlo method (=random sampling method) converges at the rate of $O(N^{-1/2})$, this method is more effective for numerical integrations.

However, many authors have reported that the quasi Monte Carlo method does not converge so fast as it is expected, if the dimension is very high. In extreme cases, it is observed to converge at the rate of $O(N^{-1/2})$, namely, exactly as slow as the Monte Carlo method. This phenomenon is often called "*the curse of dimensionality*", and it has been explained by some intuitive arguments (e.g. [15]), but no rigorous discussion has ever been made to explain the observed convergence rate, for example, $O(N^{-1/2})$ in extreme cases. Of course, even the curse of dimensionality cannot contradict with the convergence rate (1), so that it must be an intermediate or transient state, which will eventually disappear and the rate $O(N^{-1+\varepsilon})$ will appear after that.

In this paper, we tried to explain "the curse of dimensionality" in extreme cases by a rigorous probabilistic discussion for the low-discrepancy sequences generated by the Weyl transformation (= irrational rotation). In doing this, we were inspired by the following claim of Sobol' et al. ([11, 12]):

CLAIM (Sobol' et al.). *In high dimensions, the quasi Monte Carlo method is no*

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