J. Math. Kyoto Univ. (JMKYAZ)
39-2 (1999) 233-248

Generating elements for B_{dR}^{+}

By

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Introduction

Let us fix a prime number p. Then B_{dR}^{+} denotes the ring of p-adic periods of algebraic varieties defined over local (p-adic) fields as considered by J.-M. Fontaine in [Fo]. It is a topological local ring with residue field $\mathbf{C}_p$ (see the section Notations) and it is endowed with a canonical, continuous action of $G := \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$, where $\bar{\mathbf{Q}}_p$ is the algebraic closure of $\mathbf{Q}_p$ in $\mathbf{C}_p$. Let us denote by I its maximal ideal and $B_n := B_{dR}^{+}/I^n$. Then B_{dR}^{+} (and B_n for each $n \geq 1$) is canonically a $\bar{\mathbf{Q}}_p$-algebra and moreover $\bar{\mathbf{Q}}_p$ is dense in B_{dR}^{+} (and in each B_n respectively) if we consider the "canonical topology" on B_{dR}^{+} which is finer than the I-adic topology.

Let now L be any algebraic extension of $\mathbf{Q}_p$ contained in $\bar{\mathbf{Q}}_p$ and $G_L := \mathrm{Gal}(\bar{\mathbf{Q}}_p/L)$. In [I-Z], the authors described all the algebraic extensions of $K := \mathbf{Q}_p^{ur}$ such that L is dense in $(B_n)^{G_L}$ for some n or in $(B_{dR}^{+})^{G_L}$. Let us formulate this problem in a different way. For two commutative topological rings $A \subset B$, a subset $M \subset B$ will be called a "generating set" if $A[M]$ is dense in B.

Definition 0.1. Let $A \subset B$ be commutative topological rings, then we define "the generating degree", $gdeg(B/A) \in \mathbf{N} \cup \infty$ to be

$$gdeg(B/A) := \min\{|M|, \text{ where } M \text{ is a generating set of } B/A\}$$

where $|M|$ denotes the number of elements of M if M is finite and ∞ if M is not finite.

Then the problem *Is L dense in $(B_{dR}^{+})^{G_L}$?* can be formulated as *Is $gdeg((B_{dR}^{+})^{G_L}/L)$ zero?* For example Theorem 0.1 of [I-Z] can be restated as:

Theorem 0.1. *If L is not a deeply ramified extension of K then*

$$gdeg((B_n)^{G_L}/L) = 0 \quad \text{for all } n \quad \text{and} \quad gdeg(B_{dR}^{+})^{G_L}/L) = 0.$$

A characterization of deeply ramified extensions L of K satisfying $gdeg((B_{dR}^{+})^{G_L}/L) = 0$ is obtained in [I-Z], Theorem 0.2. As not all deeply ramified extensions of K have this nice property, [I-Z] left open the problem of describing $(B_n)^{G_L}$ for all n and $(B_{dR}^{+})^{G_L}$, for a general deeply ramified extension L. The first part of this paper (section 2) supplies such a description, namely we prove

Communicated by Prof. K. Ueno, November 25, 1997
Revised September 22, 1998