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A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension

By

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1. Introduction

We consider the Cauchy problem for nonlinear Klein-Gordon equations

$$(1.1)\qquad \begin{cases} (\Box + 1)u = F(u, u_t, u_x, u_{tx}, u_{xx}) & \text{in } (0,\infty)\times \mathbf{R}, \\ u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where $\Box = \partial_t^2 - \partial_x^2$, $u_t = (\partial/\partial t)u$, $u_x = (\partial/\partial x)u$, etc. We suppose that the nonlinear term F is a smooth function in its arguments around the origin and satisfies

$$(1.2)\qquad F(\lambda) = O(|\lambda|^m) \quad \text{near } \lambda = 0$$

with some integer $m \geq 2$, where $\lambda = (u, u_t, u_x, u_{tx}, u_{xx})$. For simplicity, we assume that $f, g \in C_0^\infty(\mathbf{R})$. ε is a small positive parameter.

There are many studies on global existence of solutions to this type of equations, and here we recall some known results briefly. For n-space dimensional cases with nonlinear terms of mth degree as in (1.2), Klainerman–Ponce ([11]) and Shatah ([15]) showed that if $n(m-1)^2/(2m) > 1$, then there exists a unique global solution, provided that ε is sufficiently small. This condition means that $m \geq 4$ when $n = 1$, $m \geq 3$ when $n = 2, 3, 4$ and $m \geq 2$ when $n \geq 5$. For the case $n = 3, 4$ and $m = 2$, Klainerman ([9]) and Shatah ([16]) proved independently the global existence of the solution for small ε. Klainerman used the "method of invariant norms" to get a decay estimate of the solution, which was first found to be useful in the study of nonlinear wave equations (see [8]). On the other hand, Shatah used the method of normal forms to eliminate the quadratic parts of the nonlinear terms and got the sufficient decay estimate. When $n = 2$ and $m = 2$, Georgiev–Popivanov ([4]) and Kosecki ([12]) proved the global existence of the solution, assuming that the quadratic parts of the nonlinear terms satisfy certain special conditions. For general nonlinear terms with $m = 2$ in two space dimensions, Simon–Taflin ([17]) and Ozawa–Tsutaya–Tsutsumi ([14]) proved the global existence for small ε, and showed that the solution approaches a free solution as $t \to +\infty$. In [14], they combined the methods of normal forms and of the invariant norms to get the result.

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