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Self homotopy group of the exceptional Lie group G_2

By

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1. Introduction

Let G be a connected Lie group and $\mu: G \times G \to G$ the multiplication of G. For any space A with a base point, the based homotopy set $[A, G]$ becomes a group with respect to the binary operation $\mu_*: [A, G] \times [A, G] = [A, G \times G] \to [A, G]$. Even if A is a simple space such as the sphere, it is difficult to calculate the group $[A, G]$. A general result was given by Whitehead (p. 464 of [10]):

$$\text{nil}[A, G] \leq \text{cat}\, A, \tag{1.1}$$

where nil and cat denote the nilpotency class and the Lusternik-Schnirelmann category with $\text{cat}\{*\} = 0$, respectively. In [5], we determined the group structure of $[G, G]$ and proved $\text{nil}[G, G] = 2$ when G is $SU(3)$ or $Sp(2)$. We want to study $\text{nil}[G, G]$ for other G's. Though we have very few results, it seems reasonable to set the following:

Conjecture 1.1 If G is simple, then $\text{nil}[G, G] \geq \text{rank} G$.

A weaker one is

Conjecture 1.2. If G is simple and $\text{rank} G \geq 2$, then $\text{nil}[G, G] \geq 2$, that is, $[G, G]$ is not commutative.

Let G_2 be the exceptional Lie group of rank 2. Then the purpose of this note is to prove the following which supports 1.1.

Theorem 1.3. $\text{nil}[G_2, G_2] = 3$.

Two conjectures are false in general without the assumption of simpleness of G.

Example 1.4. (1). $\text{nil}[S^3 \times S^1, S^3 \times S^1] = 1$ and $\text{nil}[U(2), U(2)] = 2$. Notice that $S^3 \times S^1$ and $U(2)$ are homeomorphic but not isomorphic.

(2). If $G = S^3 \times \cdots \times S^3$ (n times), then $\text{rank} G = n$ and $\text{nil}[G, G]$ equals 3 if $n \geq 3$

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